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MEASURES OF EFFECTIVENESS FOR SELECTED INVENTORY POLICIES UNDER STOCHASTIC PROCESSES

A THESIS

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MEASURES OF EFFECTIVENESS FOR SELECTED INVENTORY

POLICIES UNDER STOCHASTIC PROCESSES

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May 2, 1961

FOREWORD

The inspiration for this research resulted from an awareness of the importance of controlling inventories and a dissatisfaction with the present state of inventory theory. Dr. Joseph J. Moder provided encouragement in pursuing this topic. Further enthusiasm for this research grew as a consequence of my association with Mr. Joseph B. Talbird, Dr. David C. Ekey, and Dr. Harold E. Smalley in the study of inventory control in hospitals. This study was conducted under the auspices of the Engineering Experiment Station, Georgia Institute of Technology.

Dr. Moder was extremely helpful in the initial formulation of the statistical and probabilistic portions of the research. At many critical moments during this work, Dr. John MacKay and Dr. Eric R. Immel supplied important advice regarding the manipulation of probabilities; for this advice, I am extremely grateful.

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SUMMARY

The primary objective of this study is to determine measures of effectiveness for four commonly used inventory policies in which the demand and the replenishment lead time are uncontrolled variables. The secondary objective is to develop procedures for (1) the selection of optimal values of decision variables for a given inventory policy, (2) the study of the sensitivity of total relevant cost, and (3) the choosing of the best policy from among the four selected policies.

The four inventory policies selected for study are referred to as the fixed cycle inventory policy, the (s,S) inventory policy, the variable cycle inventory policy, and the combination inventory policy. The literature search suggests that these four policies are the most commonly used ones for which theoretical studies and practical applications have been reported.

The fixed cycle inventory policy, the first policy studied, requires that a replenishment order be placed at the beginning of a cycle composed of a fixed number of periods. This order is for a quantity equal to the difference between the order level and the beginning stock level. The (s,S) inventory policy, the second policy studied, requires that the stock level be observed at the beginning of a fixed number of periods. An order is placed for a quantity equal to the difference between the order level and the beginning stock level if the beginning stock level is equal to or less than the lower order level; otherwise, no order is placed. The variable cycle inventory policy, the third policy

studied, requires that a replenishment order be placed at the beginning of the first period in which the stock level is equal to or below the reorder point. This order is for a quantity equal to the difference between the order level and the stock level. The combination inventory policy, the fourth and final policy studied, requires that a replenishment order be placed at the beginning of a cycle composed of a fixed number of periods and also at the beginning of the first period within a specified number of periods in which the sum of stock level and stock on order is equal to or less than the reorder point. Both the beginning cycle and within-the-cycle replenishment orders are for a quantity equal to the difference between the order level and the sum of stock level and stock on order.

Each of these selected inventory policies is categorized according to the provision for dealing with unsatisfied demands. One condition is subject to the hypothesis that delayed fulfillment of unfilled demand may be possible; that is, back-orders are allowed. The other condition is subject to the hypothesis that all demand must be fulfilled instantaneously; that is, back-orders are not allowed.

The following assumptions are used in the study: 1. The stock level is in a steady state condition. 2. The items in stock are of a discrete nature. 3. Demand for items in stock is variable; the nature of and parameters for the demand distribution are known. 4. The replenishment lead time is variable; the nature of and parameters for the replenishment lead time distribution are known. 5. The quantity of items in the replenishment order is identical with the quantity ordered. 6. Demand for items in stock is both mutually independent and identically distributed from period to period and from cycle to cycle. 7. The replenishment lead time is both mutually independent and identically distributed from cycle to cycle. 8. The inventory policies considered are independent of other policies of the firm such as production policies and sales policies.

The stationary probabilities of the stock level at the beginning of the order cycle and at the beginning of each period within the cycle are developed in terms of controlled decision variables and uncontrolled variables. The conditional probability of transition from one stock level at the beginning of one cycle to any stock level at the beginning of the subsequent cycle does not depend upon the manner in which the stock level during the present cycle was attained. Therefore, the beginning cycle stock level is a Markov chain. The development of the beginning cycle stock level stationary probabilities requires the determination of the stock level transition probabilities. These transition probabilities are the probabilities that the stock level is equal to a specific value at the beginning of the subsequent cycle, given a specific value for the stock level at the beginning of the present cycle. The period stock level stationary probabilities are developed in terms of the beginning stock level stationary probabilities. These stock level probabilities are used as the basis for determining the measures of effectiveness.

The relevant measures of effectiveness are the probability of one or more shortages, the expected number of shortages, the expected intensity of the shortages, the expected number of items in inventory, and the expected number of replenishment orders. The expected intensity of shortages is a combined measure which weights the number of shortages by the duration of the shortages. This measure is of importance only under the hypothesis of back-orders allowed. The probability of one or more shortages during an order cycle is determined by dichotomizing the possible paths by which shortages can occur. Shortages can occur during the replenishment lead time, during the remainder of the cycle, or during both portions of the order cycle. The other measures of effectiveness are determined using the expected value formula.

The selection of values for the decision variables are developed for each of the inventory policies by iterative procedures. These procedures are developed for each of two criteria--cost minimization, given that all inventory costs are known, and a tolerated probability of a shortage at a minimal cost. This latter criterion is applicable to those situations in which all inventory costs except the cost of a shortage are known.

A procedure for the study of the sensitivity of total relevant cost is developed in order to ascertain the effects upon total relevant cost of (1) deviations in the values of the decision variables from the optimal values and (2) errors in the estimation of the cost and distribution parameters. Sensitivity curves are developed as an aid in comparing expected total relevant cost for non-optimal conditions with that for optimal conditions.

A procedure for the selection of the best inventory policy from among the four policies under study is developed. This procedure is based upon the following five inventory costs: shortage cost, carrying cost, routine ordering cost, special ordering cost, and surveillance cost.

The present study extends the analytical developments for four commonly used inventory policies. For the fixed cycle inventory policy and the (s,S) inventory policy, the present study extends the applicability of analytical developments to include conditions in which the replenishment lead time is variable. For the variable cycle inventory policy, the present study extends the applicability of analytical developments to conditions in which a replenishment order is not necessarily placed when the stock level declines to the reorder point. For the combination inventory policy, the present study extends the applicability of analytical developments to include variable demand and variable replenishment lead time for the hypothesis that back-orders are allowed. In a real inventory situation, there is little reason to expect that either the demand or the replenishment lead time are not variable.

The analytical results reported in the present study add realism to models of inventory situations. Deficiencies in these inventory models remain. The analytical development for each of the inventory policies in the present study requires that the replenishment lead time be bounded, with the analytical development for the variable cycle inventory policy requiring, in addition, that the demand during any period be bounded. It is likely in real inventory situations that both the demand and the replenishment lead time do, in fact, have an upper bound; although such a condition need not occur.

This research culminates with a conceptual framework in which decision procedures may be facilitated. The measures of effectiveness are depicted as functions of the decision variables for each of the inventory policies. If and when such a conceptual framework is transformed to a numerical basis, the results of this research can have engineering application. This numerical basis may take the form of charts, as suggested in the present study, in which the analytical expressions for the measures of effectiveness are applicable to special theoretical demand and replenishment lead time distributions. These special distributions should be typical of those distributions that are likely to occur in practice.

CHAPTER I

INTRODUCTION

Objectives of Study

The primary objective of this study is to determine relevant measures of effectiveness for four commonly used inventory policies in which the demand and the replenishment lead time are uncontrolled variables. The secondary objective is to develop decision procedures for (1) the selection of optimal values of decision variables for a given inventory policy, (2) the study of the sensitivity of total relevant cost, and (3) the choosing of the best policy from among the four selected policies.

Importance of the Study of Inventories

Inventories are necessary in the modern economy in order to fulfill at least one of the following three business motives: transactive motive, precautionary motive, and speculative motive. These motives may be defined in the following manner:

1. The transactive motive is the desire to synchronize the inflow and outflow of goods.

2. The precautionary motive is the desire for protection against uncertainty.

3. The speculative motive is the desire to profit through changes in market conditions.

The importance of inventories is indicated by the fact that in 1958 there were 34.8 billions of dollars invested in inventory items in retail and wholesale businesses. This figure represents 63.4 per cent of gross private domestic investment during 1958 (8, pp. 399-411). The costs that are directly associated with the cost of inventories are carrying costs, ordering costs or set-up costs, and shortage costs. The literature search revealed a high degree of variability in the rate of carrying cost, ranging from 6 to 35 per cent of inventory value. Applying a commonly-used rate of 25 per cent of average inventory investment, the total carrying cost in retail and wholesale business during 1958 is estimated to have been 8.7 billions of dollars. Although the literature search revealed no practical procedure for making a gross estimate, it is likely that the other two inventory costs are of significant magnitudes.

The industrial engineer is concerned with inventory problems because inventory control is important in managerial decisions. To serve the needs of management, the industrial engineer must develop decision rules such that the desired inventory objective of the firm is likely to be achieved. The industrial engineer usually limits his efforts to the transactive motive and the precautionary motive in the design of inventory control systems. The speculative motive ordinarily is considered to be a responsibility of management per se. The present study will be concerned with satisfying both the transactive motive and the precautionary motive in the development of measures of effectiveness of four commonly used inventory policies.

Selected Inventory Policies

The inventory policies selected for study here will be referred to as the fixed cycle inventory policy, the (s,S) inventory policy, the variable cycle inventory policy, and the combination inventory policy. The literature search to be reported in Chapter II suggests that these particular policies are the most commonly used ones for which theoretical studies and practical applications have been reported. Indeed, the literature search disclosed no inventory policies differing significantly from those selected for the present study.

For purposes of the mathematical treatment to follow, it is convenient to categorize each of the selected inventory policies according to two possible conditions for dealing with unsatisfied demands. One condition is subject to the hypothesis that delayed fulfillment of unfilled demands may be possible; that is back-orders are allowed. The other condition is subject to the hypothesis that all demands must be fulfilled instantaneously; that is, back-orders are not allowed.

When back-orders are allowed, any unsatisfied demands are filled (1) as soon as a subsequent replenishment order is received or (2) if the replenishment order is not large enough to fill all back-orders, then as soon as a sufficient quantity of items becomes available. Under this hypothesis, there is an infinite number of possible values for the stock level, considering unsatisfied demands as negative stock level.

When back-orders are not allowed, any unsatisfied demands either are lost or are satisfied through priority shipment. Under this hypothesis, there is a finite number of possible values for the stock level.

For the purposes of the present study, a restriction will be placed upon the upper limit of the replenishment lead time random variable for each of these four inventory policies.

Fixed Cycle Inventory Policy

Figure 1 (page 4) illustrates the fixed cycle inventory policy. The stock level is examined at the beginning of each order cycle. Each

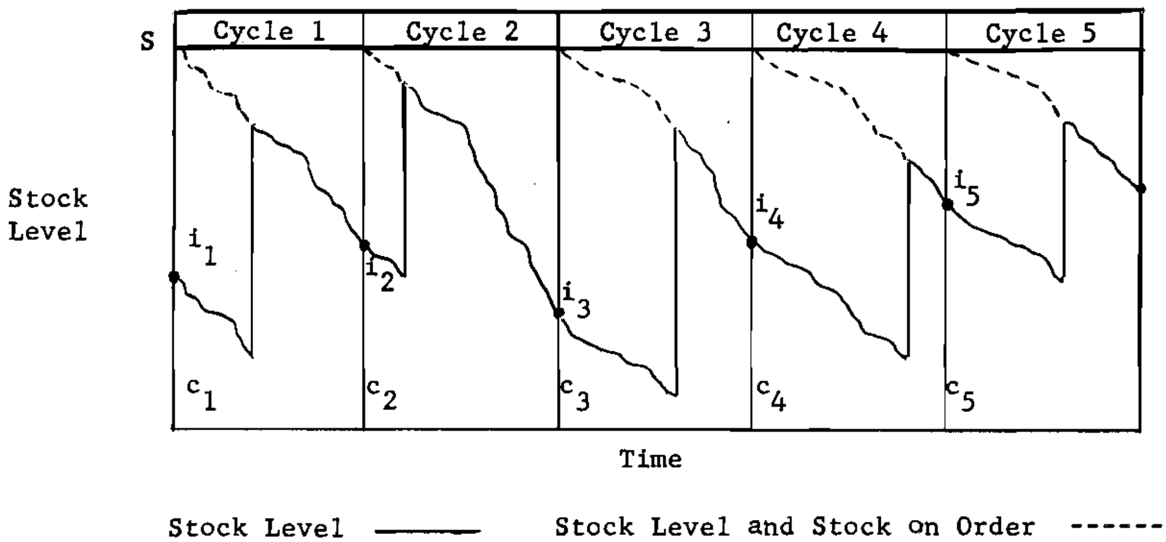

Fig. 1. Illustration of the Fixed Cycle Inventory Policy

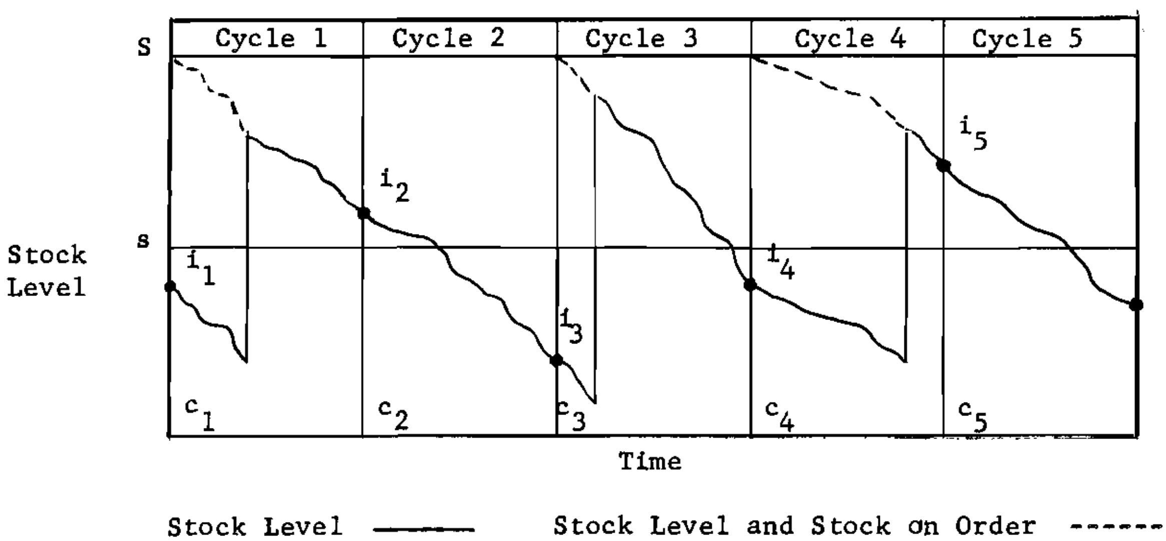

Fig. 2. Illustration of the (s,S) Inventory Policy

order cycle is of fixed duration. The stock level, i, at the beginning of the t^{th} order cycle, c_t, is denoted as i_t. A replenishment order is placed for a quantity of items equal to the difference between the order level, S, and the beginning stock level, i_t.

In analyses of the fixed cycle inventory policy, it usually is assumed in the literature that the demand, or the replenishment lead time, or both are constant. However, the present study will treat the demand and the replenishment lead time as random variables in any probability distribution.

The (s,S) Inventory Policy

Figure 2 (page 4) illustrates the (s,S) inventory policy. The stock level is examined at the beginning of each order cycle. Each order cycle is of fixed duration. A replenishment order is placed for a quantity of items equal to the difference between the (upper) order level, S, and the beginning stock level, i_t, provided the beginning stock level is equal to or less than the lower order level, s. Cycle 1 in Figure 2 illustrates the placing of a replenishment order. Cycle 2 in Figure 2 illustrates the condition when no replenishment order is placed because the beginning stock level, i_2, is greater than the lower order level, s.

In analyses of the (s,S) inventory policy, it usually is assumed in the literature that the demand, or the replenishment lead time, or both are constant. The present study will treat the demand and the replenishment lead time as random variables in any probability distribution.

The Variable Cycle Inventory Policy

Figure 3 (page 6) illustrates the variable cycle inventory policy. This inventory policy is sometimes referred to as either the fixed quantity

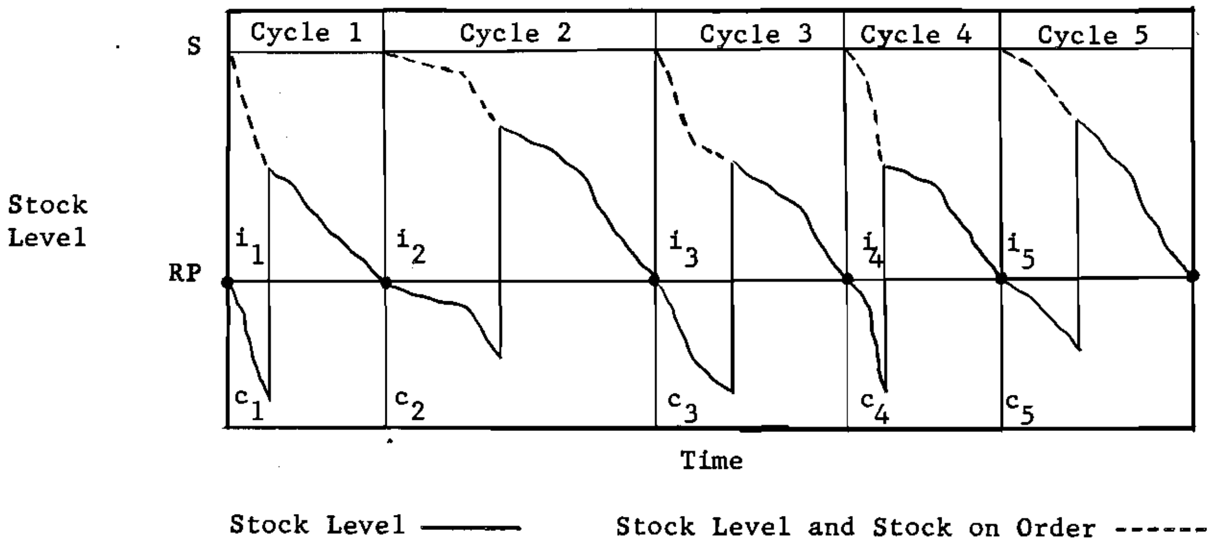

Fig. 3. Illustration of the Variable Cycle Inventory Policy

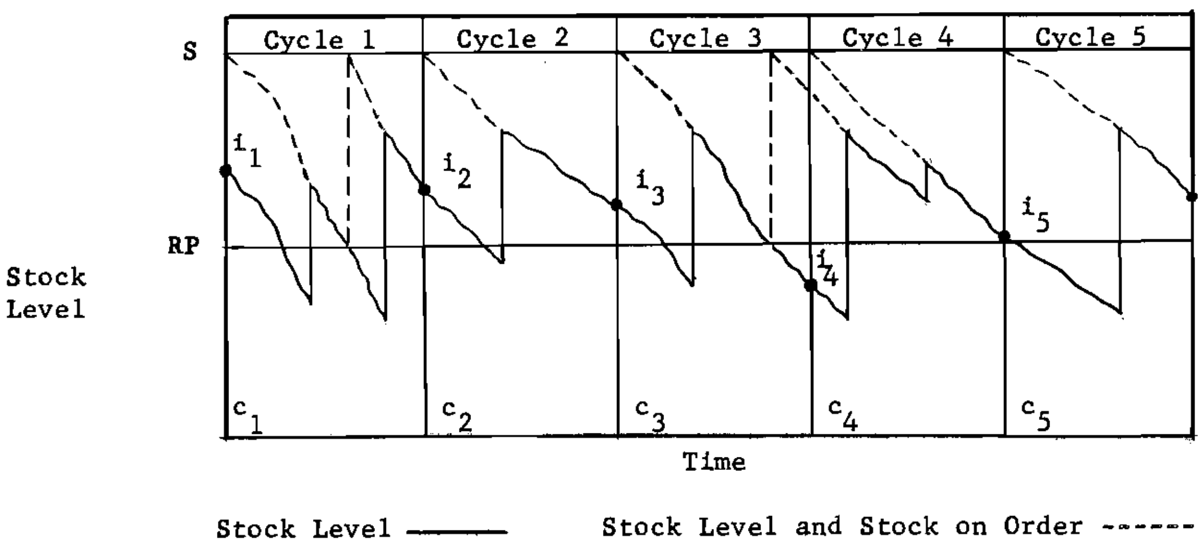

Fig. 4. Illustration of the Combination Inventory Policy

inventory policy, the economic lot size policy, or the trigger rule system. This policy requires that a replenishment order be placed for a fixed quantity of items equal to the difference between the order level, S, and the reorder point, RP. Usually, it is assumed that stock level surveillance is continuous, that a replenishment order can be placed with the vendor at any time, and that each request is for exactly one item.

The treatment in the present study will remove these assumptions. The present study will not assume that the stock level is monitored continuously, that a replenishment order can be placed at any time, nor that each request is for only one item. Without these assumptions, the stock level may be below the reorder point when the replenishment order is placed. The replenishment order is, not fixed, being equal to or greater than S-RP; and the number of periods between the placement of successive replenishment orders are variable. Consequently, a more descriptive name for this policy is the variable cycle inventory policy. Also, the present study will treat the demand and the replenishment lead time as random variables in any probability distribution.

In addition to the restriction placed upon the upper limit of the replenishment lead time, there will be a restriction upon the upper limit of the demand during any period. The effect of these two restrictions is to preclude the placement of a replenishment order before the previous replenishment order is received.

The Combination Inventory Policy

Figure 4 (page 6) illustrates the combination inventory policy. The stock level is assumed to be examined continuously within the cycle. This inventory policy requires that a replenishment order be placed at the beginning of each order cycle and also whenever the sum of stock level and stock on order is equal to or less than the reorder point, RP. At the beginning of each of the order cycles in Figure 4, a replenishment order is placed for a quantity of items equal to the difference between the order level, S, and the beginning stock level, i_t. Cycle 1 of Figure 4 illustrates a within-the-cycle replenishment order that is placed and received during the same cycle. Cycle 3 of Figure 4 illustrates a replenishment order that is placed within-the-cycle and received during the next cycle, Cycle 4.

With the combination inventory policy, it is possible that many within-the-cycle replenishment orders will be placed. Consequently, the combination inventory policy may be predominately a variable cycle inventory policy. A justification for ordering at fixed intervals is the convenience and economy of grouping similar items for ordering from the same vendor.

With the combination inventory policy, it is possible that a within-the-cycle replenishment order may be placed near the end of the ordering cycle. Therefore, the next beginning cycle replenishment order may be placed for a relatively small quantity of items. These replenishment items are the items demanded since this last within-the-cycle replenishment order was placed (refer to Cycle 3 in Figure 4).

The present study will treat a combination inventory policy which eliminates the limitations cited above. The stock level is examined at the end of equal-interval periods within the order cycle rather than continuously. A replenishment order is placed at the beginning of each order cycle, and, at most, one within-the-cyle replenishment order is placed .

Only during the first specified number of periods will the within-the-cycle replenishment order be placed. Therefore, the stock level is examined at the end of each period, either until a replenishment order is placed or until a specified number of periods has occurred. If the sum of stock level and stock on order is equal to or below the reorder point during the first specified number of periods, a within-the-cycle replenishment order will be placed. This within-the-cycle replenishment order is equal to or greater than S-RP.

Scope of Study

The present study will be concerned with the analytical determination of measures of effectiveness under the stochastic processes of variable demand and variable replenishment lead time. The inventory policies studied will be the fixed cycle, the (s,S), the variable cycle, and the combination inventory policies. The effects of unsatisfied demand are considered separately for the hypothesis of back-orders allowed and the hypothesis of back-orders not allowed.

These measures of effectiveness are suitable for the selection of values of the decision variables, for the study of the sensitivity of total relevant cost, and for the selection of the best relevant inventory policy from among these four inventory policies. This analytical formulation is applicable to the firm in which stock items of a discrete nature are ordered from an independent source. The demand for these stock items may be either internal or external.

Assumptions

The following assumptions will be used throughout the present study:

1. The stock level is in a steady state condition. Major changes in policy causing significant changes in the uncontrolled random variables do not occur. The controlled decision variables, after being initially set, are not altered.

2. The items in stock are of a discrete nature.

3. The demand for items in stock is variable; the nature of and parameters for the demand distribution are known.

4. The replenishment lead time is variable; the nature of and parameters for the replenishment lead time distribution are known.

5. The acceptable items in the replenishment order are identical with the quantity ordered. Receipts of partial orders do not occur.

6. The demand for items in stock is both mutually independent and identically distributed from period to period and from cycle to cycle.

7. The replenishment lead time is both mutually independent and identically distributed from cycle to cycle.

8. The inventory policies considered are independent of other policies of the firm such as production policies and sales policies.

Study Terminology

Various definitions are necessary for the development of mathematical models which depict relationships of the measures of effectiveness and decision variables in the four inventory policies studied.

The term "period" refers to a time interval, such as a day or a week, during which it is convenient to group successive demands for an item. The term "order cycle" refers to one or more periods, such as a month, at the beginning of which a replenishment order may be placed. The number of periods per order cycle may be either fixed or variable.

Replenishment orders are considered to be placed and received at the beginning of periods. The receipt of the items of a replenishment order in zero periods is possible. This provision implies that the items ordered may be instantaneously available. The demand for items during any one period is variable.

The stock level at the beginning of a period, within an order cycle, is equal to the stock level at the end of the previous period plus, if applicable, the replenishment order received at the beginning of the period. That is, if a quantity of items is received at the beginning of a period, these items are considered to be part of the stock level at the beginning of the period. The stock level at the end of a period is equal to the stock level at the beginning of the period minus the quantity removed from stock in response to the period demand.

Study Procedure

The pursuit of the objectives of this study utilizes a three step method of procedure. The first step is the development of stock level probabilities for the four selected inventory policies. This is accomplished in Chapters III, IV, V, and VI. The second step is the determination in Chapter VII of relevant measures of effectiveness for these inventory policies. The third step is the development of decision procedures for (1) the selection of optimal values of decision variables for a given inventory policy, (2) the study of the sensitivity of total relevant cost, and (3) the choosing of the best policy from among the four selected policies. These procedures are described in Chapter VIII.

<u>Step I: Development of stock level probabilities</u>.--As a basis for the determination of measures of effectiveness, the stationary (steady state) probabilities of the stock level at the beginning of the cycle and at

the beginning of each period within the cycle will be developed in terms of controlled decision variables and uncontrolled random variables.

The development of the stationary probabilities requires the determination of the stock level transition probabilities, which are elements of a Markov matrix.* These transition probabilities are the probabilities that the stock level is equal to a specific value at the beginning of the subsequent cycle, given a specific value for the stock level at the beginning of the present cycle. There are an infinite number of possible stock levels if back-orders are allowed. However, there are a finite number of possible stock levels if back-orders are not allowed.

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*The following statements pertaining to Markov chains are paraphrased from Parzen (20, pp. 138-139):

A stochastic process is a Markov chain if the conditional probability of transition from the present state to any other state does not depend upon the arrival path into the present state. Two states communicate if it is possible to go from one state to another state, and vice versa. Therefore, if all states in a Markov chain communicate and if a state exists with positive probability such that it can remain in the same state after one time interval, then the Markov chain is ergodic. However, if in a Markov chain it is possible to go from one state to all states after a finite number of time intervals, the Markov chain is ergodic. If a Markov chain is ergodic, the unconditional probabilities of being in a particular state tend to a unique limit after statistical equilibrium (steady state) is achieved and are designated as stationary probabilities. These stationary probabilities are independent of the initial state of the system.

The previous statements will be considered as related to the four selected inventory policies in which the stock level is examined at the beginning of each of several discrete time intervals.

The beginning cycle stock level depends upon the demand random variable, the replenishment lead time random variable, and certain controlled decision variables. Random fluctuations in the stock level result from fluctuations in the demand random variable and the replenishment lead time random variable. The successive values of these random variables are assumed to be independent and identically distributed. Therefore, the conditional probability of transition from the present state to any state does not depend upon the arrival path into the present state. Hence, the beginning cycle stock level is a Markov chain. For the fixed cycle inventory policy, the variable cycle inventory policy, and the combination inventory policy, it is possible to go from any stock level to any other stock level and to remain at the same stock level after one order cycle. For the (s,S) inventory policy, it is possible to go from one stock level to all stock levels after two order cycles. Therefore, these Markov chains are ergodic. Since the beginning cycle stock level is an ergodic Markov chain, the stationary probabilities of the beginning stock level exist and are unique.

The period stock level stationary probabilities will be obtained in terms of the beginning stock level stationary probabilities. Frequent use will be made of a theorem from probability theory which states that the probabilities corresponding to mutually exclusive sets are additive. Complex sets will be divided into simpler sets which are mutually exclusive and exhaustive for ease of mathematical treatment.

Step II: Determination of relevant measures of effectiveness.--The relevant measures of effectiveness are the probability of one or more shortages, the expected number of shortages, the expected intensity of the shortages, the expected number of items in inventory, and the expected number of replenishment orders. The expected intensity of shortages is a combined measure which weights the number of shortages by the duration of the shortages. This measure is of importance only under the hypothesis of back-orders allowed. The probability of one or more shortages during an order cycle is determined by dichotomizing the possible paths by which shortages can occur. Shortages can occur during only the replenishment lead time, during only the remainder of the cycle, or during both portions of the cycle. The other measures of effectiveness will be determined using the expected value formula.

Step III: Development of procedures for the study of inventory policies.--Iterative procedures for the selection of values of the decision variables will be developed for each of the inventory policies. These procedures will be developed for each of two criteria. The first criterion is cost minimization, given that all inventory costs are known. The second criterion is that a tolerated probability of a shortage must be obtained at a minimal cost. This latter criterion is applicable to those situations in which all inventory costs except the cost of a shortage are known.

A procedure for the study of the sensitivity of total relevant cost will be developed in order to ascertain the effects upon total relevant cost of (1) deviations in values of the decision variables from the optimal values and (2) errors in the estimation of cost and distribution parameters. Sensitivity curves will be developed as an aid in comparing expected total relevant cost for non-optimal conditions with the expected total relevant cost for optimal conditions.

A procedure for the selection of the best relevant inventory policy from among the four policies under study will be developed. This procedure will be based upon the following five inventory costs: shortage cost, carrying cost, routine ordering cost, special ordering cost, and surveillance cost.

Limitations

The analytical development of the measures of effectiveness will be for the general case. Solutions to the general formulations will be a requisite to the application of these models to particular situations.

In practice, there are hybrid instances in which some of the "customers" permit back-orders while some do not. This condition was considered to be beyond the scope of the present study.

The mathematical expressions for the measures of effectiveness determined in this study have not been verified for any real situation. Although the assumptions of the present study appear to be rational and appropriate to real inventory problems, the testing of the sufficiency of these assumptions has not been undertaken.

The objective, scope, limitations, and assumptions of this study have been stated. The selected inventory policies and the procedure for the study of these policies have been described. The literature search, reported in Chapter II, will describe those developments in inventory control theory relevant to the objectives of this study.

CHAPTER II

PRESENT STATE OF INVENTORY THEORY

Introduction

This chapter reviews the development of inventory theory and defines the bounds of existing knowledge in inventory control theory relevant to the objectives of this study.

The origins of inventory theory are found as threads in the fabric of Frederick Taylor's scientific management movement, which provided a broad stimulus to the development of analytic methods for improved solutions to management problems. While some time was to pass before these threads were drawn together to establish inventory theory as a distinct area of investigation, development significant to this area occurred as early as 1915 when F. W. Harris developed and applied an economic lot size formula at the Westinghouse Electric and Manufacturing Company (21, pp. 121, 122). This lot size formula was used for the determination of proper lot sizes, taking into account two conflicting costs. Attempts were made by F. E. Raymond in 1951 to include in one formula all factors that might affect economic lot sizes.

Prior to the paper submitted by Arrow, Harris, and Marschak (1) in 1951, most of the formulations in inventory control theory attempted to represent the problem with deterministic mathematical models. Subsequent to this stochastic treatment of inventory control theory came the theoretical papers of Dvoretzky, Kiefer, Wolkowitz (5), (6), (7) and others which will be referenced later.

Contributors to the literature usually assume that the inventory objective is cost minimization. As requisites for insuring cost minimization, knowledge of inventory costs is required; also other objectives of the firm must not be in conflict with, or constrain, any of the inventory decisions. Several authors have studied inventory problems in which the above requisites are not satisfied. If the cost of a shortage is not known, the arbitrary specification of a desired probability of a shortage is sometimes used as a criterion. This arbitrary specification generally will result in non-optimal decisions. More pressing corporate objectives sometimes require that inventory investment not exceed a particular value. The resulting inventory decision rules generally will produce non-optimal decisions. For the circumstances in which either some of the inventory costs are unknown or in which constraints are placed upon inventory decisions, authors have suggested methods in which the best possible inventory decisions, subject to the existing conditions, can be selected.

The literature may be categorized with regard to whether inventories are controlled on an aggregate basis or on an item basis. Inventories are controlled on an aggregate basis when attention is focused upon the interdependency of the items. Inventories are controlled on an item basis when attention if focused upon each item independent of other items. A brief survey of the significant references which pertain to the control of inventories on an aggregate basis follows.

Holt, Modigliani, Muth, and Simon (16, pp. 181-257) suggest approximation procedures for making inventory decisions item by item, given that such decisions are to be compatible with an aggregate constraint. These authors study the fixed cycle inventory policy and the variable cycle inventory policy. The aggregate constraint is that the total inventory investment of all products, in terms of a common unit such as dollars, must be equal to some specified value. They assume that ordering cost, carrying cost, and shortage cost are known.

Churchman, Ackoff, and Arnoff (3, pp. 255-274) determine constrained lot sizes for conditions in which warehouse space is limited, machine time is limited, or both of these limitations exist. These authors assume that ordering cost, carrying cost, and shortage cost are known.

Feeney (10) proposes generalized procedures for making inventory decisions, given feasible objectives for the inventory policy in effect. These feasible objectives may specify the attainment of various combinations of aggregate measures, such as the total number of orders to be placed, the average total dollars of inventory investment, and the tolerated number of shortages. Feeney presents a method for imputing a unique set of inventory costs from these specified objectives. If these imputed costs are inserted into applicable inventory equations for obtaining values of the decision variables such as the reorder point and the order level, the specified aggregate objectives will be attained at a minimum cost.

Welch (24) presents a specialized procedure for the variable cycle inventory policy based on a concept similar to that of Feeney. The objectives in this procedure are the attainment of specified aggregate measures for the average inventory investment and the number of total orders to be processed.

Since the present study will be concerned with the study of inventories on an item basis, a more detailed account of this category of inventory control literature, as relevant to the objectives of this study, will be presented in the following sections.

Selection of Values for Decision Variables

The significant literature relative to the selection of values for the decision variables on an item basis for the four inventory policies will be cited. The criterion used by the authors cited for the selection of values for the decision variables is cost minimization.

Fixed Cycle Inventory Policy

The decision variables to be selected for the fixed cycle inventory policy are the order level, the length of the order cycle, or both the order level and the cycle length.

Vazsonyi (23, pp. 338-345) assumes the demand to be a continuous variable and the replenishment lead time to be zero. An expression for the order level is developed under the hypothesis of back-orders not allowed by assuming that the cost of a shortage is proportional to the probability of a shortage.

Whitin (25, pp. 50-52) assumes the demand to be a random variable from a normal probability distribution. He also assumes that the standard deviation is the square root of demand. The replenishment lead time is assumed to be known and constant. A solution for approximating the optimal length of the order cycle is determined when (1) the probability of a shortage is specified, (2) the ordering cost is known, (3) and the carrying cost is known.

Gaver (14) assumes the demand to be a continuous variable and the replenishment lead time to be constant with a one cycle lag. An expression for the order level is determined for the hypothesis that back-orders are allowed and also for the hypothesis that back-orders are not allowed. Stationary probabilities are developed for the stock level at the beginning of the order cycle, for the size of the replenishment order, and for the stock deficit at the end of the order cycle. From these stationary probabilities and from the relevant costs, an optimal selection for the order level is specified. Gaver points out that the difference in the optimal order level decision values between the hypotheses of back-orders allowed and back-orders not allowed diminishes as the ratio of the shortage cost to the carrying cost increases.

Flagle, Huggins, and Roy (13, pp. 346-348) assume the demand to be a discrete variable and the replenishment lead time to be zero. Back-orders are allowed. The order level is obtained by iterative procedures. The authors assume knowledge of inventory costs, in which the cost of a shortage is proportional to the product of the number of items short and the duration of the shortage.

Sasieni, Yaspan, and Friedman (22, pp. 82-86) assume the demand to be a discrete variable and the replenishment lead time to be zero. Back-orders are not allowed. Expressions for both the order level and the cycle length are obtained by iterative procedures. The authors assume knowledge of inventory costs, in which the cost of a shortage is proportional to the number of items short.

Churchman, Ackoff, and Arnoff (3, pp. 217-223) assume the demand to be a discrete variable and the replenishment lead time to be zero. However, back-orders are allowed. Expressions for both the order level and the cycle length are obtained by iterative procedures. The authors assume knowledge of inventory costs; however, the cost of a shortage is proportional to the product of the number of shortages and the duration of the shortage.

Morse (19) assumes the demand to be a discrete variable and the replenishment lead time to be constant and equal to or less than the length of the order cycle. Morse recognizes that the class of inventory policies in which the stock level is examined at the end of an order cycle of fixed length is an example of a Markov process. The stationary probabilities for the stock level at the beginning of the cycle are determined. An iterative procedure is suggested for selecting values either for the stock level or for the cycle length, under either of the hypotheses concerning back-orders. The inventory costs are assumed to be known, with the cost of a shortage proportional to the number of items short. Although Morse does not relate any details of the formulation when both the demand and the replenishment lead time are variables, he suggests that solutions for this case could be obtained under these more complex conditions.

Arrow, Karlin, and Scarf (2) assume the demand to be a continuous variable and the replenishment lead time to be constant at a known integral multiple of the order cycle. The inventory costs are assumed to be known. The stationary probabilities for the stock level at the beginning of the order cycle are determined (1) when only one replenishment order may be outstanding and (2) when back-orders are not allowed (2, pp. 172-178). An expression for the order level is obtained in closed form for one specific demand density function. Also an expression for the order level is obtained (1) when one or more replenishment orders may be outstanding and (2) when back-orders are allowed (2, pp. 227-229). The authors assume that carrying costs and shortage costs are linear with the average inventory and the number of shortages, respectively. They point out that if the replenishment lead time is variable, the order level also can be determined. For this determination, it is necessary to obtain an appropriate compounding of the multiple-fold convolution of the order cycle demand.

Significant developments in the selection of values for the decision variables for the fixed cycle inventory policy are summarized in Table 1 (page). The application of these developments are limited to those situations in which the assumption of a known replenishment lead time is satisfied. In Chapter III the stock level stationary probabilities will be developed for the fixed cycle inventory policy under the assumptions that both the demand and the replenishment lead time are random variables.

The (s,S) Inventory Policy

The decision variables to be selected for the (s,S) inventory policy are the lower order level, s, and the order level, S; the length of the order cycle; or the order levels and the cycle length jointly.

Arrow, Harris, and Marschak (1) assume the demand to be a continuous variable and the replenishment lead time to be zero. A general expression is developed for a dynamic inventory situation which could be used for the determination of the lower order level, s, and the order level, S. A specific solution for the two order levels is developed in closed form, given a special demand density function. The inventory costs are assumed to be known; the cost of a shortage is proportional to the number of items short. This reference is a frequently-referenced paper in the stochastic treatment of inventory conditions.

Vazsonyi (23, pp. 353-359) suggests the method of retrospective simulation for the determination of the order levels, s and S, such that

Table 1. Summary of Analytical Developments for the Fixed Cycle Inventory Policy

REFERENCE	CRITERION: Costs C_I	CRITERION: Costs C_R	CRITERION: Costs C_L A	CRITERION: Costs C_L B	CRITERION: Costs C_L D	CRITERION: E	DECISION RULES: S	DECISION RULES: N	DEMAND: Variable	LEAD TIME: Constant	LEAD TIME: Variable	BACK-ORDERS: Allowed	BACK-ORDERS: Not Allowed	SS	RESULTS: Pr	RESULTS: Appr	G
Vazsonyi (23, pp. 338-345)	x	x			x		x		Continuous	Zero			x		x		0
Whitin (25, pp. 50-52)	x	x				x		x	Normal	Finite		x		x		x	≥ 0
Gaver (14)	x	x	x				x		Continuous	One Cycle Lag		x	x	x	x		0,1
Sasieni, et. al. (22, pp. 82-86)	x	x	x				x	x	Discrete	Zero			x	x	x		0
Churchman, et. al. (3, pp. 217-223)	x	x		x			x	x	Discrete	Zero		x		x	x		0

A shortage cost proportional to number of shortages.

B shortage cost proportional to the product of the number of shortages and the duration of the shortage.

D shortage cost proportional to the probability of a shortage.

E criterion of probability of a shortage.

SS steady state

Pr precise

Appr approximate

C_I carrying cost

C_R ordering cost

C_L shortage cost

G number of outstanding orders

Table 1. Summary of Analytical Developments for the Fixed Cycle Inventory Policy (continued)

REFERENCE	CRITERION						DECISION RULES		DEMAND	LEAD TIME		BACK-ORDERS		SS	RESULTS		G
	Costs					E	S	N	Variable	Constant	Variable	Allowed	Not Allowed		Pr	Appr	
	C_I	C_R	C_L														
			A	B	D												
Flagle, et. al. (13, pp. 346-348)	x	x		x			x		Discrete	Zero		x		x	x		0
Morse (19)	x	x	x				x	x	Discrete	One Cycle Lag or Less		x	x	x	x		0,1
Arrow, et, al. (2, pp. 172-178)	x	x	x				x		Continuous	Multiple of Order Cycle			x	x	x		0,1
Arrow, et. al. (2, pp. 227-229)	x	x	x				x		Continuous	Multiple of Order Cycle		x		x	x		≥ 1

A shortage cost proportional to number of shortages.

B shortage cost proportional to the product of the number of shortages and the duration of the shortage.

D shortage cost proportional to the probability of a shortage.

E criterion of probability of a shortage.

SS steady state

Pr precise

Appr approximate

C_I carrying cost

C_R ordering cost

C_L shortage cost

G number of outstanding orders

minimum cost is achieved. This method could be used in any situation, regardless of whether or not the demand and the replenishment lead time are variables and regardless of the hypothesis concerning back-orders. Since such a method is laborious and time consuming, more direct analytical procedures may be preferred.

Sasieni, Yaspan, and Friedman (22, pp. 86-92) assume the demand to be a discrete variable and the replenishment lead time to be zero. A solution is obtained by iterative procedures for the order levels, s and S, under the hypothesis that back-orders are allowed. The authors assume knowledge of inventory costs, of which the cost of a shortage is proportional to the product of the number of items short and the duration of the shortage.

Magee (17, pp. 83-86) develops expressions for approximating the order levels, s and S, under conditions of variable demand and a constant lead time. The lower order level, s, is determined by specifying the tolerated probability of a shortage during the interval composed of the order cycle and the replenishment lead time. From an intuitive relationship between the parameters, the order levels, s and S, can be determined. The author assumes that the ordering cost and carrying cost are known.

Arrow, Karlin and Scarf (2) assume the demand to be a continuous variable and the replenishment lead time to be zero. For the steady state condition, general expressions are developed for the determination of the lower order level, s, and the order level, S. The inventory costs are assumed to be known; the cost of a shortage is proportional to the number of items short. These expressions are applicable either to the hypothesis that back-orders are allowed (2, pp. 229-234) or to the hypothesis that back-orders are not allowed (2, pp. 280-285). Specific solutions, each for a special demand density function, are developed separately under these hypotheses. Other solutions for the order levels, s and S, are developed by the authors for such special conditions as single cycle inventory problems, dynamic inventory problems, and non-linear inventory costs.

Significant developments in the selection of values for the decision variables for the (s,S) inventory policy are summarized in Table 2 (page). The application of these developments are limited to those situations in which the assumption of a known replenishment lead time is satisfied. In Chapter IV, the stock level stationary probabilities will be developed under the assumptions that both the demand and the replenishment lead time are random variables.

Variable Cycle Inventory Policy

The decision variables to be selected for the variable cycle inventory policy are the order level and the reorder point. The following references assume (1) that the demand and the replenishment lead time are variables and (2) that it is possible to place a replenishment order as soon as the demand reduces the stock level to the reorder point. The latter assumption implies (1) that stock level surveillance is continuous, (2) that a replenishment order can be placed with the vendor at any time, and (3) that each demand is for exactly one item. This inventory policy is sometimes referred to as either the fixed quantity inventory policy, the economic lot size policy, or the trigger rule system.

Harling and Bramson (15) develop an expression for the probability that the stock level is positive when the replenishment order is received.

Table 2. Summary of Analytical Developments for the (s,S) Inventory Policy

REFERENCE	CRITERION					DECISION RULES			DEMAND	LEAD TIME		BACK-ORDERS		SS	RESULTS		G
	Costs				E	S	N	s	Variable	Constant	Variable	Allowed	Not Allowed		Pr	Appr	
	C_I	C_R	C_L														
			A	B													
Arrow, et. al. (1)	x	x	x		x	x		x	Continuous	Zero			x	x	x		0
Sasieni, et. al. (22, pp. 86-92)	x	x		x		x		x	Discrete	Zero		x		x	x		0
Magee (17, pp. 83-86)	x	x			x	x		x	Discrete	Finite		(1)	(1)	x		x	≥0
Arrow, et. al. (2, pp. 229-234)	x	x	x			x		x	Continuous	Zero		x	x	x	x		0

A shortage cost proportional to number of shortages.

B shortage cost proportional to the product of the number of shortages and the duration of the shortage.

(1) not considered

E criterion of probability of a shortage.

SS steady state

Pr precise

Appr approximate

C_I carrying cost

C_R ordering cost

C_L shortage cost

G number of outstanding orders

This probability is designated as the level of protection. For any desired level of protection the reorder point can be specified.

Ekey, Talbird, and Newberry, (9) develop expressions for the probability of one or more shortages during the lead time. The results of these expressions are presented graphically for Poisson densities over a limited range of the distribution means. A reorder point can be selected which corresponds to any desired probability of a shortage during an interval in which one or more replenishment orders are placed.

Clark and Rowe (4) state that the independent selection of the order level and the reorder point will result in non-optimal solutions of the parameters. A procedure for approximating the reorder point and the order level is presented, given a specified probability of a shortage. The distribution of demand during lead time is developed from empirical data. The authors assume that the ordering cost and the carrying cost are known.

Holt, Modigliani, Muth, and Simon (16, pp. 220-243) develop under the hypothesis of back-orders allowed, an expression for determining the reorder point when the order quantity is specified. The authors assume that the inventory costs are known. Expressions are developed in which shortage costs are (1) proportional to the number of shortages, (2) proportional to the product of the number of shortages and the duration of the shortages, and (3) proportional to the maximum interval of time the item is back-ordered.

Fetter and Dalleck (12, pp. 13-18) develop procedures for selecting both the reorder point and the order level, under either of the hypotheses concerning back-orders. The demand and the replenishment lead time distributions are combined into a single distribution of demand during lead time. The authors assume knowledge of inventory costs in which the cost of a shortage is proportional to the number of shortages.

Morse (18, pp. 139-156) analyzes, by queueing theory, inventory policies in which a fixed quantity of items is ordered when the sum of stock level and stock on order is equal to the reorder point. Measures of effectiveness are developed for the special inventory situation under the hypothesis of back-orders not allowed when (1) the reorder point is equal to the order level minus one item, (2) the demand is a Poisson density, and (3) the replenishment lead time is a negative exponential density (18, pp. 139-146). There is a possibility of more than one replenishment order being outstanding at any one time. Formulas for approximating the order level are developed. The inventory costs are assumed to be known. Morse suggests procedures for the analysis of hybrid inventory situations in which some of the "customers" permit back-orders while some do not.

Arrow, Karlin, and Scarf (2, pp. 285-288) consider the case in which the demand density is Poisson and the replenishment lead time is negative exponential, under the hypothesis that back-orders are allowed. Expressions are formulated for determining the reorder point and the order level when knowledge of inventory costs is complete and when knowledge of inventory costs is partially incomplete. For the latter case, the carrying cost and the ordering cost are assumed to be known, and the probability of a shortage is specified. Only one replenishment order is outstanding at any one time. The authors remove some of these restrictions (2, pp. 307-318) in a later section as follows: (1) the demand density may be any density, (2) back-orders either may be allowed or not allowed, and (3) the number of replenishment orders outstanding at any time is unlimited.

The above-cited references assume that the replenishment order may be placed as soon as the demand reduces the stock level to the reorder point. However, the references pertaining to the (s,S) inventory policy by Arrow, Karlin, and Scarf, (2, pp. 229-234) and (2, pp. 280-285), are of the type considered in the present study which are designated as the variable cycle inventory policy. These references do not assume that a replenishment order can be placed as soon as the stock level is reduced to the order level; however, these references do assume the replenishment lead time to be zero.

Significant developments in the selection of values for the decision variables for the variable cycle inventory policy are summarized in Table 3 (page 31). The application of these developments are limited to those situations in which either (1) the replenishment lead time is zero or (2) a replenishment order is placed immediately when the sum of stock level and stock on order is reduced to the reorder point. In Chapter V the stock level stationary probabilities will be developed under the assumptions that both the demand and the replenishment lead time are random variables and that a replenishment order is placed at the end of the first period in which the sum of stock level and stock on order is equal to or below the reorder point.

Combination Inventory Policy

The only reference encountered which relates to an inventory policy composed of characteristics of the fixed cycle policy and the variable

Table 3. Summary of Analytical Developments for the Variable Cycle Inventory Policy

REFERENCE	CRITERION						DECISION RULES		DEMAND	LEAD TIME		BACK-ORDERS		F	SS	RESULTS		G
	Costs					E	S	RP	Variable	Constant	Variable	Allowed	Not Allowed			Pr	Appr	
	C_I	C_R	C_L															
			A	B	C													
Harling, et. al. (15)						(2)		x	Poisson		Normal	(1)	(1)	x	x	x		≥ 0
Ekey, et. al. (9)						x		x	Poisson		Poisson	(1)	(1)	x	x	x		≥ 0
Clark, et. al. (4)	x	x				x	x	x	(3)		(3)	x	x	x	x		x	≥ 0
Holt, et. al. (16, pp. 220-243)	x	x	x		x	x		x	Continuous		Continuous	x		x	x		x	≥ 0

A shortage cost proportional to number of shortages.

B shortage cost proportional to the product of the number of shortages and the duration of the shortage.

C shortage cost proportional to maximum interval of time the item is back-ordered.

(1) not considered.

(2) criterion is probability of a stock-out.

(3) demand during lead time.

E criterion of probability of a shortage.

F immediate placement of order is possible.

SS steady state

Pr precise

Appr approximate

C_I carrying cost

C_R ordering cost

C_L shortage cost

G number of outstanding orders

Table 3. Summary of Analytical Developments for the Variable Cycle Inventory Policy (continued)

REFERENCE	CRITERION						DECISION RULES		DEMAND	LEAD TIME		BACK-ORDERS		F	SS	RESULTS		G
	Costs					E	S	RP	Variable	Constant	Variable	Allowed	Not Allowed			Pr	Appr	
	C_I	C_R	A	B	C													
Fetter, et. al. (12, pp. 13-18)	x	x	x			x	x	x	(3)		(3)	x	x	x	x	x		≥ 0
Morse (18, pp. 139-156)	x	x	x				x	S-1	Poisson		Negative Exponential		x	x	x	x		≥ 0
Arrow, et. al. (2, pp. 255-258)	x	x	x			x	x	x	Poisson		Negative Exponential	x		x	x	x		0,1
Arrow, et. al. (2, pp. 307-318)	x	x	x						Continuous		Negative Exponential	x	x		x	x		≥ 0

A shortage cost proportional to number of shortages.

B shortage cost proportional to the product of the number of shortages and the duration of the shortage.

C shortage cost proportional to maximum interval of time the item is back-ordered.

(1) not considered.

(2) criterion is probability of a stock-out.

(3) demand during lead time.

E criterion of probability of a shortage.

F immediate placement of order is possible

SS steady state

Pr precise

Appr approximate

C_I carrying cost

C_R ordering cost

C_L shortage cost

G number of outstanding orders

cycle policy is one by Whitin (25, p. 22). He states that the General Services Administration operates under this type of inventory policy. No analytical decision rules have been reported for this inventory policy. In Chapter VI the stock level stationary probabilities will be developed for this policy, under the assumptions that the demand and the replenishment lead time are random variables.

Sensitivity Analysis

The precision required in the selection of optimal values of the decision variables may be determined by a study of the sensitivty of the total relevant cost to deviations from the optimal values of these decision variables. Also the precision required for data collection may be determined by a study of the sensitivity of the total relevant cost to deviations in the estimation of cost or distribution parameters.

Flaggle, Huggins and Roy (13, pp. 350-362) study the effects of deviations in the values of the decision variables from their optimal values upon the total relevant cost. These authors also study the effects of deviations in the values of constants, such as costs and distribution means, upon total relevant cost. These studies are for two deterministic inventory situations.

Fetter and Dalleck (1, pp. 22-26) mention the need for studies of the sensitivity of total relevant cost in inventory situations. Such studies will promote an understanding of the consequences of errors in the estimation of cost and distribution parameters.

A more detailed procedure for the study of parameter sensitivity for stochastic inventory situations will be suggested in Chapter VIII.

Selection of Inventory Policies

The selection of the best relevant inventory policy may be as important as the selection of optimal values of the decision variables for a given policy; yet little work has been reported in the development of criteria for use in the selection of practical inventory policies.

Magee (17, pp. 94-96) suggests some qualitative features regarding the degree of control which may be obtained in the fixed cycle inventory policy and in the variable cycle inventory policy. Whitin (25, p. 23) suggests certain costs which he considers to be determinants in the selection of an inventory policy. The costs which he lists are routine ordering cost, special ordering cost, carrying cost, and surveillance cost.

Arrow, Karlin, and Scarf (2, pp. 110-178) specify a mathematical basis for the selection of optimal inventory policies. These optimal inventory policies are selected on the basis of the mathematical properties of the analytical expressions for the ordering cost, carrying cost, and shortage cost. The estimation of these analytical cost expressions and the derivatives of the cost expressions appear to be insurmountable because of limitations in present cost estimation techniques.

The only reference cited for the quantitative selection of inventory policies is based upon the mathematical properties of the expressions for ordering cost, carrying cost, and shortage cost. Since such cost expressions cannot be practically estimated, such a selection basis has limited practical utility. In Chapter VIII a procedure will be developed which has practical utility. The inventory cost expressions are linear with the appropriate measure of effectiveness. Also, in addition to the costs usually considered in contemporary literature, a special ordering cost and a surveillance cost is included. Implicitly, the selection of one inventory policy in preference to another is based on the combination of the characteristics of the inventory policies and the specific costs associated with the inventory situation.

Discussion of Results

The significant theoretical developments relevant to the objectives of this study have been referenced.

For the fixed cycle inventory policy and the (s,S) inventory policy (summarized in Tables 1 and 2), the lack of any analytical development for conditions in which the lead time is variable is readily apparent. Generally, precise analytical treatment of conditions in which more than one replenishment order is outstanding have been neglected. One author does permit multiple replenishment orders to be outstanding, but he requires that the lead time for these orders be constant and occur in multiples of the order cycle. For these two inventory policies, the present study will extend the applicability of analytical developments to conditions in which the replenishment lead time is variable; but, at most, only one replenishment order is outstanding. The analytical development for inventory situations in which the joint occurrence of variable lead time and multiple replenishment orders is beyond the scope of the present study.

For the variable cycle inventory policy (summarized in Table 3), it is assumed that a replenishment order may be placed immediately when the stock level declines to the reorder point. Such an assumption is not always valid. The present study will extend the applicability of analytical developments to conditions in which an order is not necessarily placed when the stock level declines to the reorder point. An upper bound is placed upon the demand during any period. The analytical development for these inventory situations in which the demand during any period is unbounded is beyond the scope of the present study.

For the combination policy, no analytical developments have been cited. The analytical development in the present study permits variable demand and variable replenishment lead time, but this development is restricted to the conditions in which back-orders are allowed. The development for conditions in which back-orders are not allowed is beyond the scope of the present study.

The literature search revealed no significant developments either in the study of the sensitivity of total relevant cost or in the choosing of a practical inventory policy. The present study will develop procedures for the study of the sensitivity of total relevant cost to deviations in the values of the decision variables and to errors in cost and distribution parameters. Also procedures will be developed for the choosing of a practical inventory policy.

The next chapter is the first of four chapters in which the stock level stationary probabilities will be developed. These stock level probabilities will be used in the determination of the measures of effectiveness. On the basis of these measures of effectiveness, the decision procedures will be developed.

CHAPTER III

STOCK LEVEL PROBABILITIES FOR THE FIXED CYCLE INVENTORY POLICY

Introduction

The objective of this chapter is to develop mathematical expressions for the stock level probabilities in terms of relevant controlled decision variables characteristic of the fixed cycle inventory policy and uncontrolled random variables. The fixed cycle inventory policy requires that a replenishment order be placed at the beginning of a cycle composed of a fixed number of periods. This order is for a quantity equal to the difference between the order level, S, and the beginning stock level, i. The relevant controlled decision variables are the order level, S, and the number of periods, N, in the order cycle. The uncontrolled random variables are the demand and the replenishment lead time. These stock level probabilities will be developed separately for the hypotheses of back-orders allowed and back-orders not allowed.

Under the fixed cycle inventory policy, it usually is assumed that the demand, the replenishment lead time, or both are constant. The treatment of the fixed cycle inventory policy in this present study will consider the demand and the replenishment lead time as random variables obtained from any probability distribtuion. The basic assumptions listed and discussed in Chapter I will be used as a basis for the analytical development of the fixed cycle inventory policy considered in this chapter. In addition, the replenishment lead time is assumed to be less than the number of periods, N, in the order cycle.

Back-Orders Allowed

The objective of this section is to develop mathematical expressions for the stock level probabilities under the hypothesis of back-orders allowed. These expressions will be developed in terms of the demand and the replenishment lead time probability distributions; the order level, S; and the number of periods, N, in the order cycle.

When back-orders are allowed, unsatisfied demands are deferred either until the next replenishment order is received or until sufficient stock is available. Under the fixed cycle inventory policy, a replenishment order is placed at the beginning of each order cycle. The quantity ordered depends upon either the difference between the order level, S, and available stock or upon the sum of order level, S, and unsatisfied demand. That is,

$$\text{replenishment order} = \begin{cases} S - \text{available stock for } i > 0, \\ \\ S + \text{unsatisfied demand for } i \leq 0. \end{cases} \qquad (1)$$

Beginning Stock Level Stationary Probabilities

The probabilities that the beginning stock level is equal to i, a(i), after statistical equilibrium is attained, will be determined by solving an infinite Markov transition matrix equation.

<u>Markov transition matrix equation</u>.--A transition from any beginning cycle stock level, i, to any stock level, j, at the beginning of the subsequent fixed order cycle, for $i \leq S$ and $j \leq S$, is possible as a consequence of the ordering rule. Therefore, the transition matrix, $M = (m_{ij})$, is ergodic; and the stationary probabilities exist and are unique. The stationary probabilities will be determined by solving the following matrix equations:

$$\underline{a} = \underline{a}\, M \,, \tag{2}$$

$$\text{where } \underline{a} = [a(S),\ a(S-1),\ \ldots,\ a(0),\ \ldots], \tag{2a}$$

$$\text{and } M = \begin{bmatrix} m_{S,S} & m_{S,S-1} & \cdots & m_{S,0} & \cdots \\ m_{S-1,S} & m_{S-1,S-1} & \cdots & m_{S-1,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{0,S} & m_{0,S-1} & \cdots & m_{0,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{2b}$$

In the infinite matrix, M, m_{ij} is the probability that the stock level is j items at the beginning of the subsequent cycle, given that the stock level was i items at the beginning of the present cycle.

Stock level transition probabilities.--If demand during the cycle does not exceed S items, all demand is satisfied before the subsequent replenishment order is placed. However, if the demand during the order cycle is greater than S items, there is unsatisfied demand (negative stock level at the beginning of the subsequent cycle). This unsatisfied demand causes the replenishment order at the beginning of the subsequent cycle to be in excess of S items by an amount equal to the unsatisfied demand. This carry-over demand is satisfied independently of the subsequent cycle demand. The replenishment order is always equivalent to the demand which occurred during the previous cycle. Therefore, the beginning stock level for the subsequent cycle is dependent upon the demand during the previous cycle but independent of the beginning stock level, i. That is,

$$m_{ij} = P\{D(N)=S-j\} \qquad \text{for } i \leq S \text{ and } j \leq S. \tag{3}$$

The stationary probabilities, $a(i)$, that the stock level is equal to i at the beginning of the order cycle will be determined from expression (2). Since the rows of the matrix are independent of i (identical), expression (3) represents the stationary probabilities and may be written as

$$a(j) = m_{ij} \qquad \text{for } i \leq S \text{ and } j \leq S. \tag{4}$$

By a change in subscript notation,

$$a(i) = m_{ji} \qquad \text{for } i \leq S \text{ and } j \leq S. \tag{5}$$

Period Stock Level Stationary Probabilities

The stationary probabilities that the stock level is equal to h prior to the demand in period r will be denoted by $a_r(h)$. These period stock level probabilities will be developed from the period stock level conditional stationary probabilities, $a_r(h|x)$, which correspond to N mutually exclusive and exhaustive sets, in conjunction with the probability of the specific replenishment lead time, $P\{X=x\}$. That is,

$$a_r(h) = \sum_{x=0}^{N-1} [a_r(h|x)]P\{X=x\} \qquad \text{for } 1 \leq r \leq N \text{ and } h \leq S. \tag{6}$$

<u>Period stock level conditional stationary probabilities</u>.--The conditional stationary probabilities that the stock level is equal to h prior to the demand in period r, given that the replenishment lead time is x, are denoted by $a_r(h|x)$. These conditional probabilities will be developed in terms of the beginning stock level stationary probabilities, $a(i)$ from the relationship of the period, r, within the order cycle to the

replenishment lead time, $X=x$. In this development, consider the cases corresponding to the two mutually exclusive and exhaustive sets: $r < x+1$ and $r \geq x+1$. It will be necessary to determine all of the possible combinations of beginning stock level, i, and also of the demand, $D(r-1)$, prior to period r so that the quantity of items in stock is equal to h prior to the demand in period r.

I. Case I: $r < x+1$.--A specific path in which the stock level is equal to h occurs if the beginning stock level is equal to i, for $i \leq S$, and if the demand, $D(r-1)$, during the $r-1$ periods is equal to $i-h$. The probability of this specific path is

$$a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq x+1 \leq N,\ i \leq S, \quad \text{and } h \leq S. \tag{7}$$

Figure 5 (page 42) illustrates this general path.

The value of the beginning stock level can be any of the possible values from h to S inclusive. Therefore,

$$a_r(h|x) = \sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq r < x+1 \leq N \text{ and } h \leq S. \tag{8}$$

II. Case II: $r \geq x+1$.--The period stock level conditional probabilities will be developed for this case by considering the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the mutually exclusive and exhaustive sub-sets: $x=0$ and $1 \leq x \leq N-1$.

A. Sub-case I: $x=0$.--The replenishment order is received in zero time periods; therefore the stock level prior to the demand during the first period is instantaneously S items. Since $S-i$, for $i \leq S$, items

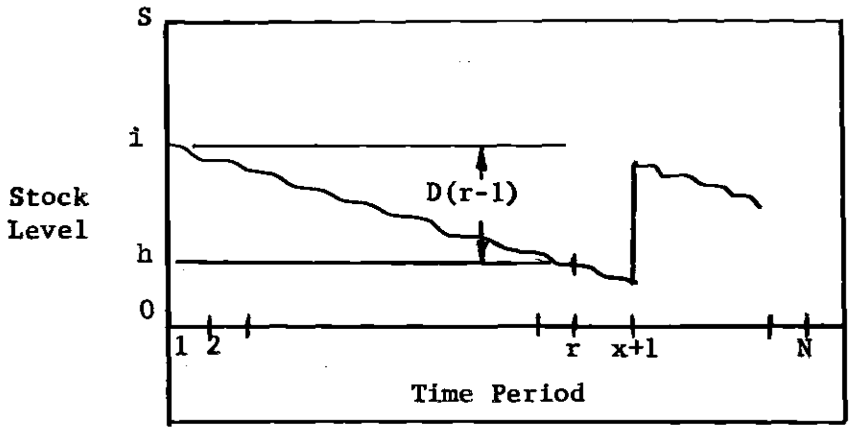

Fig. 5. General Stock Level Path for the Fixed Cycle Inventory Policy, Back-Orders Allowed, r < x+1

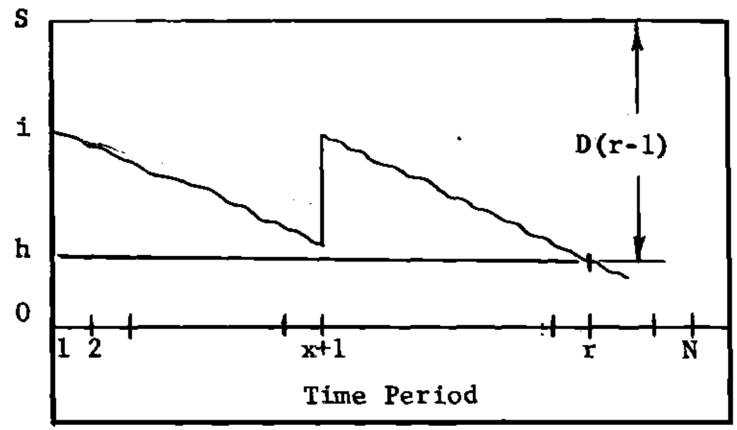

Fig. 6. General Stock Level Path for the Fixed Cycle Inventory Policy, Back-Orders Allowed, r ≥ x+1

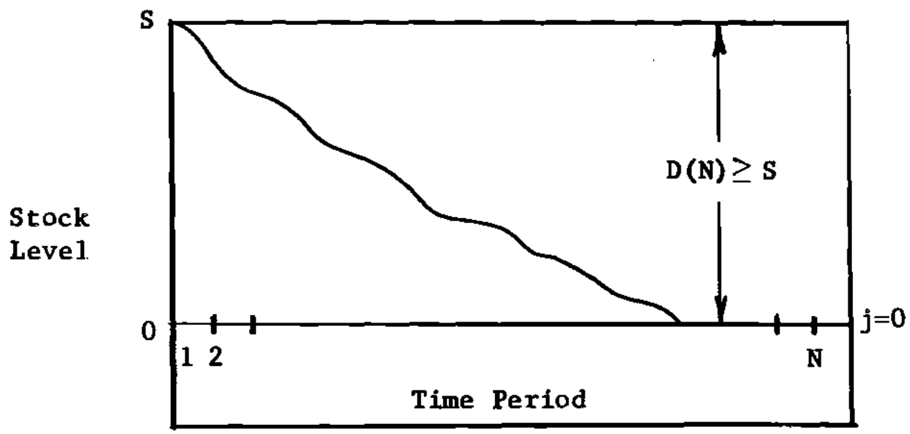

Fig. 7. General Stock Level Transition Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, x=0 and j=0

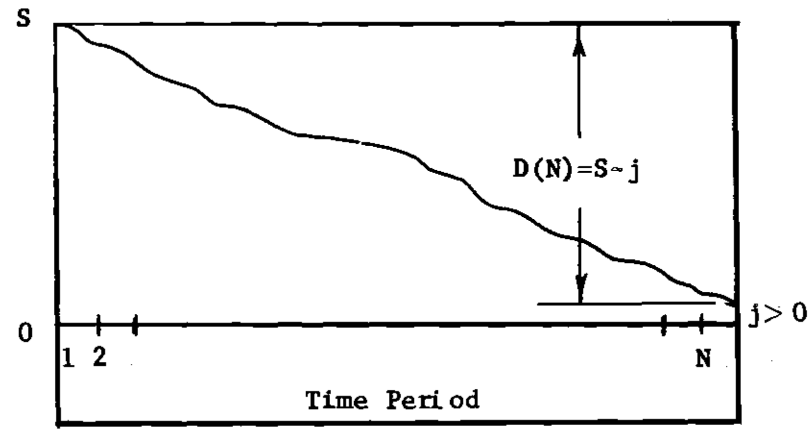

Fig. 8. General Stock Level Transition Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, x=0 and j > 0

are ordered at the beginning of the order cycle and received instantaneously, $i+(S-i)=S$. Therefore,

$$a_1(h|0) = \delta_{hS}{}^{*}. \tag{9}$$

B. Sub-Case II: $1 \leq x \leq N-1$.--There are effectively S items available from which the demand during the r-1 periods is made. A negative stock level indicates that the demand during the r-1 periods is greater than S. If the demand, $D(r-1)$, during the r-1 periods is for S-h items, the quantity of items available prior to the demand during period r is $S-(S-h)$, or h items. Therefore,

$$a_r(h|x) = P\{D(r-1)=S-h\} \quad \text{for } 1 < x+1 \leq r \leq N \quad \text{and } h \leq S. \tag{10}$$

Figure 6 (page 42) illustrates this general path.

III. Summary of period stock level conditional stationary probabilities.--Expressions (8), (9), and (10) are summarized as follows:

$$a_r(h|x) = \begin{cases} \displaystyle\sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} & \text{for } 1 \leq r < x+1 \leq N \\ & \quad \text{and } h \leq S, \\ \delta_{hS} & \text{for } 1=r=x+1 \leq N, \\ P\{D(r-1)=S-h\} & \text{for } 1 < x+1 \leq r \leq N \\ & \quad \text{and } h \leq S. \end{cases} \tag{11}$$

${}^{*}\delta_{hS}$ is the Kroneker delta symbol which is unity if $h=S$ and zero if $h \neq S$.

Period stock level unconditional stationary probabilities.--The period stock level unconditional stationary probabilities, $a_r(h)$, can be obtained by inserting expression (11) into expression (6).

The beginning stock level stationary probabilities and the period stock level stationary probabilities developed in this section will be used as the basis for determining the measures of effectiveness for the fixed cycle inventory policy under the hypothesis that back-orders are allowed.

Back-Orders Not Allowed

The objective of this section is to develop expressions for the stock level probabilities under the hypothesis of back-orders not allowed. These expressions will be in terms of the demand and the replenishment lead time probability distributions; the order level, S; and the number of periods, N, in the order cycle.

When back-orders are not allowed, demand in excess of available stock is lost if not satisfied instantaneously by priority action. Under the fixed cycle inventory policy a replenishment order is placed at the beginning of each order cycle. The quantity ordered depends upon only the difference between the order level, S, and available stock. That is,

$$\text{replenishment order} = S - \text{available stock}. \tag{12}$$

Beginning Stock Level Stationary Probabilities

The probabilities that the beginning stock level is equal to i, a(i), after statistical equilibrium is attained, can be determined by solving a finite Markov transition matrix equation.

Markov transition matrix equation.--A transition from any beginning cycle stock level, i, to any stock level, j, at the beginning of the subsequent fixed order cycle, for $0 \leq i \leq S$ and $0 \leq j \leq S$, is possible as a consequence of the ordering rule. Therefore, the transition matrix, $M=(m_{ij})$, is ergodic; and the stationary probabilities exist and are unique. These stationary probabilities can be determined by solving the following matrix equation:

$$\underline{a} = \underline{a}\, M, \tag{13}$$

$$\text{where } \underline{a} = [a(S),\ a(S-1),\ \ldots,\ a(0)], \tag{13a}$$

$$\text{and } M = \begin{bmatrix} m_{S,S} & m_{S,S-1} & \cdots & m_{S,0} \\ m_{S-1,S} & m_{S-1,S-1} & \cdots & m_{S-1,0} \\ \cdots & \cdots & \cdots & \cdots \\ m_{1,S} & m_{1,S-1} & \cdots & m_{1,0} \\ m_{0,S} & m_{0,S-1} & \cdots & m_{0,0} \end{bmatrix}. \tag{13b}$$

In the finite matrix, M, m_{ij} is the probability that the stock level is j items at the beginning of the subsequent cycle, given that the stock level was i items at the beginning of the present cycle.

Stock level transition probabilities.--The stock level transition probabilities, m_{ij}, will be developed from the stock level conditional transition probabilities, $m_{ij}|x$, corresponding to N mutually exclusive and exhaustive sets, in conjunction with the probability of the specific replenishment lead time. That is,

$$m_{ij} = \sum_{x=0}^{N-1} (m_{ij}|x)P(X=x). \tag{14}$$

I. Stock level conditional transition probabilities.--The stock level conditional transition probability element, $m_{ij}|x$, is the probability that the stock level is equal to j at the beginning of the subsequent cycle, given that the stock level was i items at the beginning of the present cycle and that the replenishment lead time is x. The possible values for the replenishment lead time, X=x, and the beginning of the subsequent cycle stock level, j, are divided into four mutually exclusive and exhaustive sets. The replenishment lead time is either zero or positive; and the beginning stock level of the subsequent cycle, j, is either zero or positive.

For zero replenishment lead time, the stock level instantaneously attains the value of S.

For a positive replenishment lead time, there are two possible paths:

1. Demand, D(x), during the replenishment lead time may be equal to or greater than the beginning stock level, i. In this case all demand in excess of the beginning stock level is lost. Figure 9 (page 49) illustrates this general path of the stock level during the replenishment lead time.

2. However, demand, D(x), during the replenishment lead time may be less than the beginning stock level, i. Figure 10 (page 49) illustrates this general path of the stock level during the replenishment lead time.

The four cases corresponding to the mutually exclusive and exhaustive sets are as follows:

Case I: Zero replenishment lead time ($x=0$) and zero stock level at the beginning of the subsequent cycle ($j=0$).

Case II: Zero replenishment lead time ($x=0$) and a positive stock level at the beginning of the subsequent cycle ($j > 0$).

Case III: Positive replenishment lead time ($x > 0$) and zero stock level at the beginning of the subsequent cycle ($j=0$).

Case IV: Positive replenishment lead time ($x > 0$) and a positive stock level at the beginning of the subsequent cycle ($j > 0$).

A. Case I: $x=0$ and $j=0$.--Since the replenishment lead time is zero, S items of stock are instantaneously available. A transition to a zero supply at the beginning of the subsequent cycle occurs if the demand, $D(N)$, during the N periods of the cycle is equal to or greater than S. Therefore,

$$m_{ij}|x = P\{D(N) \geq S\} \qquad \text{for } 0 \leq i \leq S,\ j=0, \quad \text{and } x=0. \qquad (15)$$

Figure 7 (page 42) illustrates this general path.

B. Case II: $x=0$ and $j > 0$.--Since the replenishment lead time is zero, S items of stock are instantaneously available. A transition to a positive supply of j items, for $1 \leq j \leq S$, occurs if the demand, $D(N)$, during the N periods of the cycle is equal to $S-j$. Therefore,

$$m_{ij}|0 = P\{D(N)=S-j\} \qquad \text{for } 0 \leq i \leq S \quad \text{and } 1 \leq j \leq S. \qquad (16)$$

Figure 8 (page 42) illustrates this general path.

C. Case III: $x > 0$ and $j=0$.--Consider the two general paths corresponding to the mutually exclusive and exhaustive sets which terminate with $j=0$: $D(x) \geq i$ and $D(x) < i$.

1. Path I: $D(x) \geq i$.--When demand during the replenishment lead time is equal to or greater than the beginning stock level, i, the quantity available prior to the receipt of the replenishment order is zero items. Since the replenishment order is for S-i items, the quantity available when the replenishment order is received is S-i items. Consequently, demand, $D(N-x)$, during the latter part of the cycle must be equal to or greater than S-i if $j=0$. Hence, the probability of Path I occurring is

$$P\{D(x) \geq i\}P\{D(N-x) \geq S-i\} \qquad \text{for } 0 \leq i \leq S,\ j=0, \text{ and } 1 \leq x \leq N-1. \tag{17}$$

Figure 9 (page 49) illustrates this general path.

2. Path II: $D(x) < i$.--If the quantity of items available prior to the receipt of the replenishment order is denoted as h, then the quantity of items in stock after the replenishment order is received is h+S-i. From this intermediate stock level of h+S-i items, the transition to a zero supply at the beginning of the subsequent cycle occurs if demand, $D(N-x)$, during the latter portion of the cycle is equal to or greater than h+S-i. Therefore, the probability of this particular path for a specific value of h, for $1 \leq h \leq i$, is

$$P\{D(x)=i-h\}P\{D(N-x) \geq h+S-i\} \qquad \text{for } 0 \leq i \leq S,\ j=0, \ 1 \leq x \leq N-1, \text{ and } 1 \leq h \leq i. \tag{18}$$

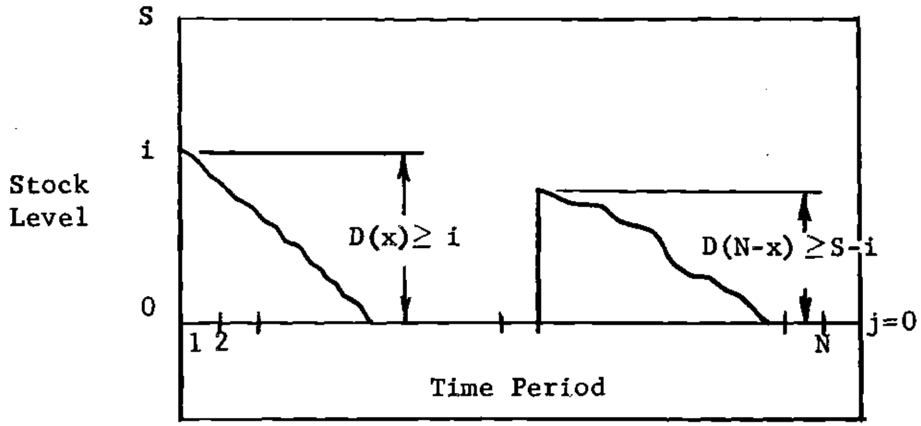

Fig. 9. General Stock Level Transition Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $x > 0$, $j=0$, and $D(x) \geq i$

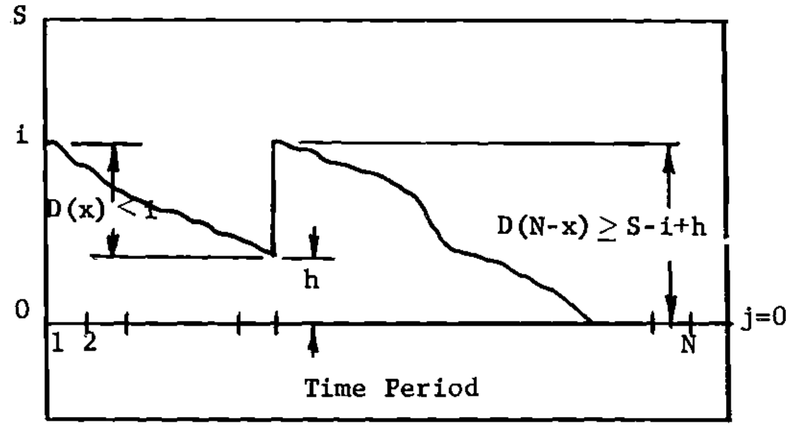

Fig. 10. General Stock Level Transition Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $x > 0$, $j=0$, and $D(x) < i$

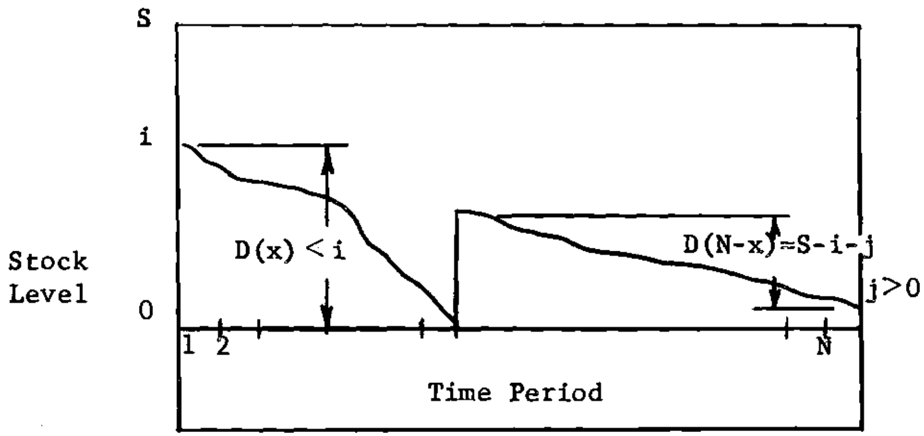

Fig. 11. General Stock Level Transition Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $x > 0$, $j > 0$, and $D(x) < i$

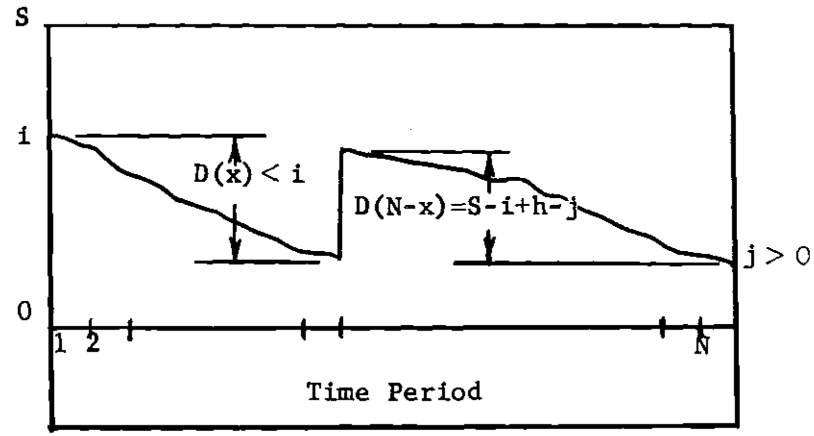

Fig. 12. General Stock Level Transition Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $x > 0$, $j > 0$, and $D(x) < i$

Figure 10 (page 49) illustrates this particular path.

Since h is the stock level prior to the receipt of the replenishment order, the possible values of h are from 1 to i inclusive. Therefore, the probability of Path II occurring is

$$\sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x) \geq h+S-i\} \qquad \text{for } 0 \leq i \leq S,\ j=0, \text{ and } 1 \leq x \leq N-1. \tag{19}$$

3. Summary of Case III.--The probability for Case III is obtained by combining expressions (17) and (19) as follows:

$$m_{ij}|x = P\{D(x) \geq i\}P\{D(N-x) \geq S-i\} + \sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x) \geq h+S-i\}$$

$$\text{for } 0 \leq i \leq S,\ j=0, \text{ and } 1 \leq x \leq N-1. \tag{20}$$

D. Case IV: $x > 0$ and $j > 0$.--Consider the two general paths corresponding to the mutually exclusive and exhaustive sets which terminate with $j > 0$: $D(x) \geq i$ and $D(x) < i$.

1. Path I: $D(x) \geq i$.--When the demand during the replenishment lead time is equal to or greater than the beginning stock level, i, the quantity available prior to the receipt of the replenishment order is zero items. The replenishment order is S-i items, therefore, the quantity available when the replenishment order is received is S-i items. Consequently, the demand, D(N-x), during the latter portion of the cycle must equal S-i-j if the supply at the beginning of the subsequent cycle is j. Therefore, the probability of Path I occurring is

$$P\{D(x) \geq i\}P\{D(N-x)=S-i-j\} \qquad \text{for } 0 \leq i \leq S,\ 1 \leq j \leq S, \text{ and } 1 \leq x \leq N-1. \tag{21}$$

Figure 11 (page 49) illustrates this general path.

2. Path II: $D(x) < i$.--If the quantity of items available just prior to the receipt of the replenishment order is denoted as h, then the quantity available after the replenishment order is received is h+S-i. From this intermediate stock level of h+S-i items, demand, D(N-x), during the latter portion of the order cycle must equal h+S-i-j if the supply at the beginning of the subsequent cycle is j. Therefore, the probability of this particular path for a specific value of h, for $1 \leq h \leq i$, is

$$P\{D(x)=i-h\}P\{D(N-x)=h+S-i-j\} \qquad \text{for } 0 \leq i \leq S,\ 1 \leq j \leq S,\ 1 \leq x \leq N-1, \text{ and } 1 \leq h \leq i. \tag{22}$$

Figure 12 (page 49) illustrates this general path.

Since h is the stock level just before the replenishment order is received, the possible values of h are any of the values from 1 to i inclusive. Therefore, the probability of Path II occurring is

$$\sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x)=h+S-i-j\} \qquad \text{for } 0 \leq i \leq S,\ 1 \leq j \leq S, \text{ and } 1 \leq x \leq N-1. \tag{23}$$

3. Summary of Case IV.--The probability for Case IV is obtained by combining expressions (21) and (23) as follows:

$$m_{ij}|x = P\{D(x) \geq i\}P\{D(N-x)=S-i-j\} + \sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x)=h+S-i-j\} \tag{24}$$

$$\text{for } 0 \leq i \leq S,\ 1 \leq j \leq S, \text{ and } 1 \leq x \leq N-1.$$

E. <u>Summary of stock level conditional transition probabilities</u>.--Expressions (15), (16), (20), and (24), are summarized as follows:

$$m_{ij}|x = \begin{cases} P\{D(N) \geq S\} & \text{for } 0 \leq i \leq S,\ j=0, \text{ and } x=0; \\ P\{D(N)=S-j\} & \text{for } 0 \leq i \leq S,\ 1 \leq j \leq S, \text{ and } x=0; \\ P\{D(x) \geq i\}P\{D(N-x) \geq S-i\} + \sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x) \geq h+S-i\} & \text{for } 0 \leq i \leq S,\ j=0, \text{ and } 1 \leq x \leq N-1; \\ P\{D(x) \geq i\}P\{D(N-x)=S-i-j\} + \sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x)=h+S-i-j\} & \text{for } 0 \leq i \leq S,\ 1 \leq j \leq S, \text{ and } 1 \leq x \leq N-1. \end{cases} \tag{25}$$

II. <u>Stock level unconditional transition probabilities</u>.--The stock level unconditional transition probability element is obtained by inserting expression (25) into expression (14). Therefore,

$$m_{ij} = \begin{cases} P\{D(N)=S-j\}P\{x=0\} + \sum_{x=1}^{N-1}\Bigg[P\{D(x) \geq i\}P\{D(N-x)=S-i-j\} \\ \quad + \sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x)=S-i+h-j\}\Bigg]\Bigg[P\{X=x\}\Bigg] \\ \qquad\qquad \text{for } 0 \leq i \leq S \text{ and } 1 \leq j \leq S, \\ \\ P\{D(N) \geq S\}P\{X=0\} + \sum_{x=1}^{N-1}\Bigg[P\{D(x) \geq i\}P\{D(N-x) \geq S-i\} \\ \quad + \sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x) \geq S-i+h\}\Bigg]\Bigg[P\{X=x\}\Bigg] \\ \qquad\qquad \text{for } 0 \leq i \leq S \text{ and } j=0. \end{cases} \tag{26}$$

The stationary probabilities for the stock level at the beginning of the order cycle can be determined from expression (13) utilizing the transition probabilities of expression (26).

Period Stock Level Stationary Probabilities

The stationary probabilities that the stock level is equal to h prior to the demand in period r will be denoted by $a_r(h)$. These period stock level probabilities can be obtained from the period stock level conditional stationary probabilities, $a_r(h|x)$, corresponding to N mutually exclusive and exhaustive sets, in conjunction with the probability of the specific replenishment lead time. That is,

$$a_r(h) = \sum_{x=0}^{N-1} [a_r(h|x)]P\{X=x\} \qquad \text{for } 1 \leq r \leq N \text{ and } h \leq S. \tag{27}$$

Period stock level conditional stationary probabilities.--The conditional stationary probabilities that the stock level is equal to h prior to the demand in period r, given that the replenishment lead time is x, are denoted by $a_r(h|x)$. These conditional probabilities will be developed in terms of the beginning stock level probabilities, a(i), from the relationship of the period, r, within the cycle to the replenishment lead time, X=x. In this development, consider the cases corresponding to the three mutually exclusive and exhaustive sets: $r < x+1$, $r=x+1$, and $r > x+1$.

I. Case I: $r < x+1$.--The period stock level conditional probabilities will be developed by considering the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: h=0 and $1 \leq h \leq S$.

A. Sub-case I: h=0.--A specific path in which the stock level prior to the demand in period r is equal to zero occurs if the beginning stock level is equal to i and if the demand, D(r-1), during the r-1 periods is equal to or greater than i. Therefore, the probability of this specific path is

$$a(i)P\{D(r-1) \geq i\} \qquad \text{for } 1 \leq r < x+1 \leq N \text{ and } 0=i \leq S. \tag{28}$$

Figure 13 (page 55) illustrates this general path.

The beginning stock level can be any of the possible values from 0 to S inclusive. Therefore,

$$a_r(h|x) = \sum_{i=0}^{S} a(i)P\{D(r-1) \geq i\} \tag{29}$$

$$\text{for } 1 \leq r < x+1 \leq N \text{ and } h=0.$$

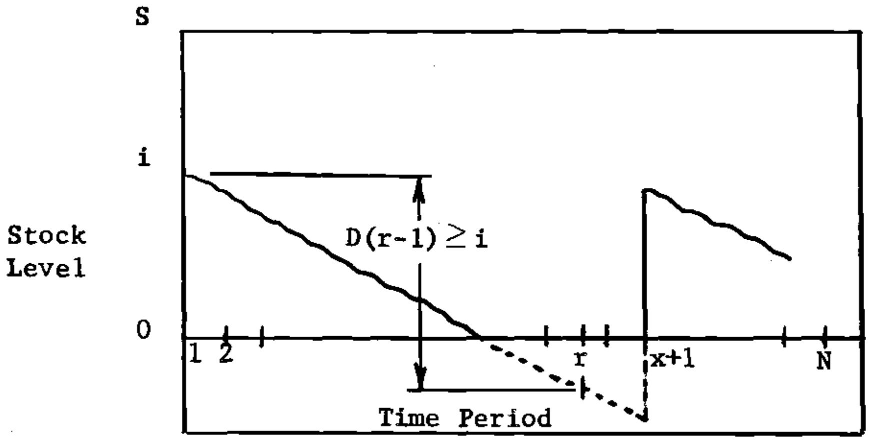

Fig. 13. General Stock Level Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $r < x+1$ and $h=0$

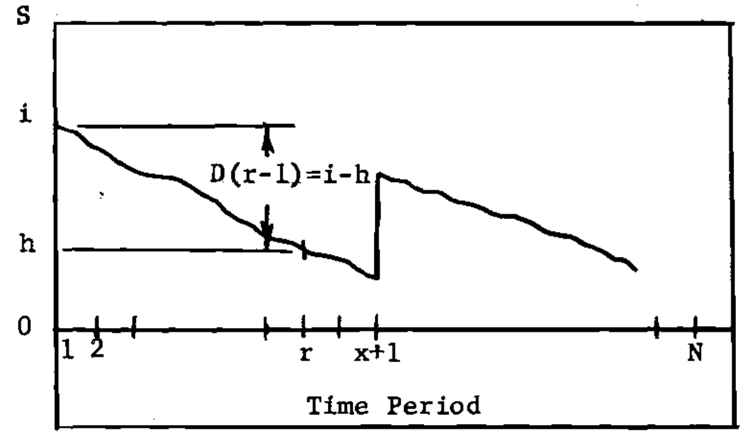

Fig. 14. General Stock Level Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $r < x+1$ and $1 \leq h \leq S$

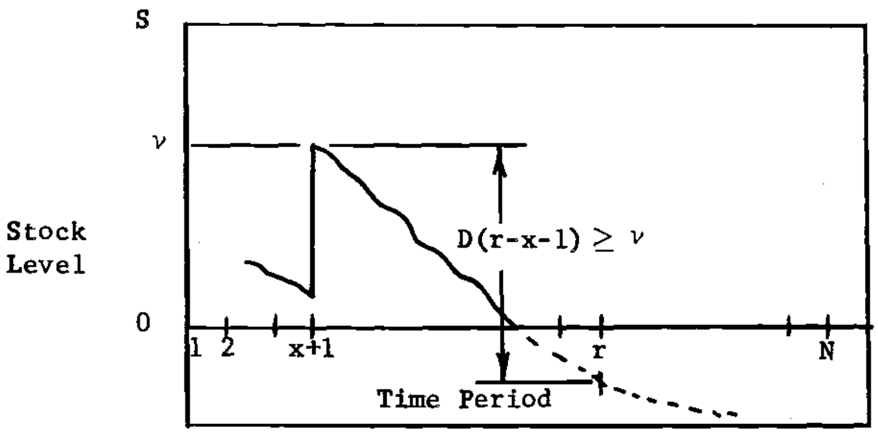

Fig. 15. General Stock Level Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $r \geq x+1$ and $h=0$

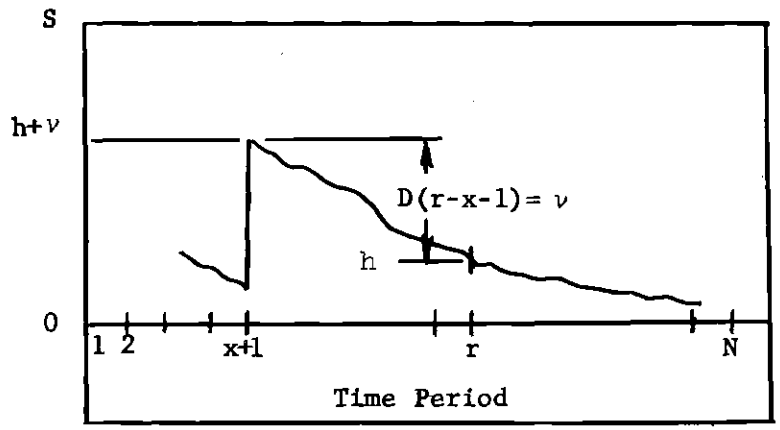

Fig. 16. General Stock Level Path for the Fixed Cycle Inventory Policy, Back-Orders Not Allowed, $r \geq x+1$ and $1 \leq h \leq S$

B. Sub-case II: $1 \leq h \leq S$.--A specific path in which the stock level prior to the demand in period r is equal to h occurs if the beginning stock level is equal to i, for $0 \leq i \leq S$, and if the demand, $D(r-1)$, during the r-1 periods is equal to i-h. Therefore, the probability of this specific path is

$$a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq r < x+1 \leq N \text{ and } 1 \leq h \leq i \leq S. \tag{30}$$

Figure 14 (page 55) illustrates this general path.

The beginning stock level can be any of the possible values from h to S inclusive. Therefore,

$$a_r(h|x) = \sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq r < x+1 \leq N \text{ and } 1 \leq h \leq S. \tag{31}$$

II. Case II: $r=x+1$.--The period stock level conditional probabilities will be developed by considering the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $x=0$ and $1 \leq x \leq N-1$.

A. Sub-case I: $x=0$.--If the replenishment order is received in zero time periods, the stock level prior to the demand in the first period is instantaneously S items. Since S-i items, for $0 \leq i \leq S$, are ordered at the beginning of the order cycle and received in zero periods, the stock level at the beginning of the first period is i+(S-i), or S. Therefore,

$$a_1(h|0) = \delta_{hS} \ . \tag{32}$$

B. <u>Sub-case II: $1 \leq x \leq N-1$</u>.--For the other cases of $1 \leq r=x+1 \leq N$, there is the possibility that the demand during the replenishment lead time is greater than the beginning stock level. The conditional cumulative probability distribution is useful in obtaining an expression for this case. The stock level prior to the demand in period x+1 is equal to or less than some specified value, say h, for $0 \leq h \leq S$, if (1) the beginning stock level is equal to or greater than S-h items (causing a replenishment order of h items or less to be placed) and if (2) the demand, D(x), during the replenishment lead time is equal to or greater than S-h units. An order for zero items is considered meaningful.

Algebraically, the previous argument is as follows. If the beginning stock level is i, for $S \geq i \geq S-h$, then S-i items are ordered. If demand exists during the replenishment lead time such that $D(x) \geq S-h$, then the stock level just prior to the receipt of the replenishment order is max (0, i-D(x)). If $D(x) \geq S-h$, there are at most i-(S-h) items available when the order is received. By hypothesis i-(S-h) is non-negative. Therefore, if Y denotes the number of items available after the replenishment order is received, then

$$Y \leq [i-(S-h)] + (S-i), \tag{33}$$

$$\text{or } Y \leq h. \tag{34}$$

The following cumulative probability distributions are defined:

$$A(S-h-1) = \sum_{i=0}^{S-h-1} a(i) \text{ and } A_r(h|x) = \sum_{i=0}^{h} a_r(i|x). \tag{35}$$

From the above argument,

$$A_{x+1}(h|x) = [1-A(S-h-1)]P\{D(x) \geq S-h\} \quad \text{for } 1 < x+1 \leq N \text{ and } 0 \leq h \leq S. \tag{36}$$

By the definition of a discrete probability,

$$a_{x+1}(h|x) = A_{x+1}(h|x) - A_{x+1}(h-1|x). \tag{37}$$

Therefore,

$$a_{x+1}(h|x) = [1-A(S-h-1)]P\{D(x) \geq S-h\} - [1-A(S-h)]P\{D(x) \geq S-h+1\} \quad \text{for } 1 < x+1 \leq N \text{ and } 0 \leq h \leq S. \tag{38}$$

III. Case III: $r > x+1$.--The period stock level conditional probabilities for this case will be obtained in terms of the period stock level probabilities when the replenishment order is received, $a_{x+1}(h|x)$. Consider the sub-cases corresponding to the two mutually exclusive and exhaustive sub-sets obtained by dichotomizing this set into $h=0$ and $1 \leq h \leq S$.

A. Sub-case I: $h=0$.--A specific path in which the stock level prior to the demand in period r is equal to zero occurs if the stock level after the replenishment order is received is ν and if the demand, $D(r-x-1)$, during the $r-(x+1)$ periods is equal to or greater than ν. Therefore, the probability of this specific path is

$$[a_{x+1}(\nu|x)]P\{D(r-x-1) \geq \nu\} \quad \text{for } 1 \leq x+1 < r \leq N, \; 0 \leq \nu \leq S, \text{ and } h=0. \tag{39}$$

Figure 15 (page 55) illustrates this general path.

The stock level when the replenishment order is received can be any of the possible values from 0 to S inclusive. Therefore,

$$a_r(h|x) = \sum_{\nu=0}^{S} [a_{x+1}(\nu|x]P\{D(r-x-1) \geq \nu\} \tag{40}$$

$$\text{for } 1 \leq x+1 < r \leq N \text{ and } h=0.$$

B. Sub-case II: $1 \leq h \leq S$.--A specific path in which the stock level prior to the demand in period r is equal to h occurs if the stock level after the replenishment order is received is $h+\nu$ and if the demand, $D(r-x-1)$, during the $r-(x+1)$ periods is ν. Therefore, the probability of this specific path is

$$[a_{x+1}(h+\nu|x]P\{D(r-x-1)=\nu\} \qquad \text{for } 1 \leq x+1 < r \leq N, \quad 0 \leq h+\nu \leq S, \text{ and } 1 \leq h \leq S. \tag{41}$$

Figure 16 (page 55) illustrates this general path.

The stock level when the replenishment order is received can be any of the possible values from h to S inclusive. Therefore,

$$a_r(h|x) = \sum_{\nu=0}^{S-h} [a_{x+1}(h+\nu|x)]P\{D(r-x-1)=\nu\} \tag{42}$$

$$\text{for } 1 \leq x+1 < r \leq N \text{ and } 1 \leq h \leq S.$$

IV. Summary of period stock level conditional stationary probabilities.--Expressions (29), (31), (32), (38), (40), and (42) are summarized as follows:

$$a_r(h|x) = \begin{cases} \sum_{i=0}^{S} a(i)P\{D(r-1) \geq i\} & \text{for } 1 \leq r < x+1 \leq N \text{ and } h=0, \\ \sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} & \text{for } 1 \leq r < x+1 \leq N \text{ and } 1 \leq h \leq S, \\ \delta_{hS} & \text{for } 1=r=x+1 \leq N, \\ [1-A(S-h-1)]P\{D(x) \geq S-h\} - [1-A(S-h)]P\{D(x) \geq S-h+1\} & \text{for } 1 < r=x+1 \leq N \text{ and } 0 \leq h \leq S, \\ \sum_{\nu=0}^{S} [a_{x+1}(\nu|x)]P\{D(r-x-1) \geq \nu\} & \text{for } 1 \leq x+1 < r \leq N \text{ and } h=0, \\ \sum_{\nu=0}^{S-h} [a_{x+1}(h+\nu|x)]P\{D(r-x-1)=\nu\} & \text{for } 1 \leq x+1 < r \leq N \text{ and } 1 \leq h \leq S. \end{cases} \quad (43)$$

Period stock level unconditional stationary probabilities.--The period stock level unconditional stationary probabilities, $a_r(h)$, can be obtained by inserting expression (43) into expression (27).

The beginning stock level stationary probabilities and the period stock level stationary probabilities developed in this section will be used as the basis for determining the measures of effectiveness for the fixed cycle inventory policy under the hypothesis that back-orders are not allowed.

Results

The expressions for the stock level probabilities for the fixed cycle inventory policy have been developed under the hypothesis of back-orders allowed and the hypothesis of back-orders not allowed. The results of this chapter will be used in Chapter VII in the determination of the measures of effectiveness required to attain the primary objective of the study.

CHAPTER IV

STOCK LEVEL PROBABILITIES FOR THE (s,S) INVENTORY POLICY

Introduction

The objective of this chapter is to develop mathematical expressions for the stock level probabilities in terms of relevant controlled decision variables characteristic of the (s,S) inventory policy and uncontrolled random variables. The (s,S) inventory policy requires that a replenishment order be placed if the beginning stock level is equal to or less than the lower order level, s. This order is placed at the beginning of a cycle composed of a fixed number of periods for a quantity equal to the difference between the order level, S, and the beginning stock level, i. The relevant controlled decision variables are the order level, S; the lower order level, s; and the number of periods, N, in the order cycle. The uncontrolled random variables are the demand and the replenishment lead time. These stock level probabilities will be developed separately for the hypotheses of back-orders allowed and back-orders not allowed.

Under the (s,S) inventory policy, it usually is assumed that the demand, the replenishment lead time, or both are constant. The treatment of the (s,S) inventory policy in this present study will consider the demand and the replenishment lead time as random variables obtained from any probability distribution. The assumptions listed and discussed in Chapter I will be used as a basis for the analytical development of the (s,S) inventory policy considered in this chapter. In addition, the replenishment lead time is assumed to be less than the number of periods, N, in the order cycle.

Back-Orders Allowed

The objective of this section is to develop expressions for the stock level probabilities under the hypothesis of back-orders allowed. These expressions will be in terms of the demand and the replenishment lead time probability distributions; the order level, S; the lower order level, s; and the number of period, N, in the order cycle.

When back-orders are allowed, unsatisfied demands are deferred until sufficient stock is available. Under the (s,S) inventory policy a replenishment order is placed at the beginning of each order cycle if the beginning stock level, i, is equal to or less than s; otherwise, no replenishment order is placed. The replenishment quentity ordered depends upon either the difference between the order level, S, and available stock or upon the sum of order level, S, and unsatisfied demand. That is,

$$\text{replenishment order} = \begin{cases} 0 & \text{for } s < i \leq S \\ S - \text{available stock} & \text{for } 0 < i \leq s \\ S + \text{unsatisfied demand} & \text{for } i \leq 0. \end{cases} \tag{1}$$

Beginning Stock Level Stationary Probabilities

The probabilities that the beginning stock level is equal to i, a(i), after statistical equilibrium is attained, can be determined by solving an infinite Markov transition matrix equation.

Markov transition matrix equation.--A transition from any beginning cycle stock level, i, to any stock level, j_2*, at the beginning of the second order cycle, for $i \leq S$ and $j_2 \leq S$, is possible as a consequence of the (s,S) ordering rule. Therefore, the transition matrix, $M=(m_{ij})$, is ergodic; and the stationary probabilities exist and are unique. These stationary probabilities can be determined by solving the following matrix equation:

$$\underline{a} = \underline{a}\, M, \tag{2}$$

where $\underline{a} = [a(S),\ a(S-1),\ \ldots,\ a(s),\ a(s-1),\ \ldots,\ a(0),\ \ldots],$ (2a)

$$\text{and } M = \begin{bmatrix} m_{S,S} & m_{S,S-1} & \cdots & m_{S,s} & \cdots & m_{S,0} & \cdots \\ m_{S-1,S} & m_{S-1,S-1} & \cdots & m_{S-1,s} & \cdots & m_{S-1,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{s,S} & m_{s,S-1} & \cdots & m_{s,s} & \cdots & m_{s,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{0,S} & m_{0,S-1} & \cdots & m_{0,s} & \cdots & m_{0,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{2b}$$

In the infinite matrix, M, m_{ij} is the probability that the stock level is j items at the beginning of the subsequent cycle, given that the stock level was i items at the beginning of the present cycle.

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*The subscript on j_2 indicates the second consecutive order cycle.

Stock level transition probabilities.--In the development of the stock level transition probabilities, consider the cases corresponding to the two mutually exclusive and exhaustive sets: $s < i \leq S$ and $i \leq s$.

I. Case I: $s \leq i \leq S$.--No replenishment order is placed. Consider the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $i < j$ and $j \leq i$, for $s < i \leq S$ and $j \leq S$.

A. Sub-case I: $i < j$.--Since no replenishment order is placed, it is impossible that the beginning stock level, j, of the subsequent cycle is greater than the beginning stock level, i, of the present cycle. Therefore,

$$m_{ij} = 0 \qquad \text{for } s < i < j \leq S. \tag{3}$$

B. Sub-case II: $j \leq i$.--Any stock level, j, of the subsequent cycle, for $j \leq i$, can be attained if the demand, $D(N)$, during the entire cycle is equal to $i-j$. That is,

$$m_{ij} = P\{D(N)=i-j\} \qquad \text{for } s < i \leq S \text{ and } j \leq i. \tag{4}$$

II. Case II: $i \leq s$.--There are effectively S items from which total demand during the order cycle is made since (1) back-orders are allowed and (2) all replenishment orders are received during the order cycle in which the order is placed. If the demand during the replenishment lead time exceeds the supply, as many of the unsatisfied demands as possible are satisfied when the replenishment order is received. If there are any unsatisfied demands at the end of the order cycle, they are considered as a negative inventory stock level. When $i \leq s$, the beginning stock level for the subsequent order cycle is dependent only upon the demand during the present order cycle and S. Therefore,

$$m_{ij} = P\{D(N)=S-j\} \qquad \text{for } j \leq S \text{ and } i \leq s. \tag{5}$$

III. Summary of stock level transition probabilities.--The stock level transition probabilities are summarized from expressions (3), (4), and (5) as follows:

$$m_{ij} = \begin{cases} 0 & \text{for } i < j \leq S \text{ and } s < i \leq S, \\ P\{D(N)=i-j\} & \text{for } i \geq j \text{ and } s < i \leq S, \\ P\{D(N)=S-j\} & \text{for } j \leq S \text{ and } i \leq s. \end{cases} \tag{6}$$

The stationary probabilities for the stock level at the beginning of the order cycle can be determined from expression (2) utilizing the transition probabilities of expression (6).

Period Stock Level Stationary Probabilities

The probabilities that the stock level is equal to h prior to the demand in period r are denoted by $a_r(h)$. These probabilities will be determined by combining the probabilities of two mutually exclusive and exhaustive sets. That is,

$$\begin{aligned} &P\{\text{stock level}=h \text{ prior to demand in period } r\} \\ &\quad = P\{\text{stock level}=h \text{ prior to demand in period } r,\ s < i \leq S\} \\ &\quad\quad + P\{\text{stock level}=h \text{ prior to demand in period } r,\ i \leq s\} \\ &\quad\quad\quad \text{for } 1 \leq r \leq N \text{ and } h \leq S. \end{aligned} \tag{7}$$

The probabilities that the stock level prior to the demand during period r is equal to h and that the stock level at the beginning of the cycle was greater than s, but not in excess of S, are denoted by $a_{r1}(h)$. The probabilities that the stock level prior to the demand during period r is equal to h and that the stock level at the beginning of the cycle was equal to or less than s are denoted by $a_{r2}(h)$. Therefore,

$$a_r(h) = a_{r1}(h) + a_{r2}(h) \qquad \text{for } 1 \leq r \leq N \text{ and } h \leq S. \tag{8}$$

Development of $a_{r1}(h)$.--A specific path in which the stock level is equal to h prior to the demand in period r occurs if the beginning stock level is equal to i, for $s < i \leq S$, and if the demand, $D(r-1)$, during the r-1 periods is equal to i-h. The probability of this specific path is

$$a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq r \leq N,\ s < i \leq S, \text{ and } h \leq i. \tag{9}$$

The number of possible paths depend upon the value of h. Consider the cases corresponding to the two mutually exclusive and exhaustive sets: $s < h \leq S$ and $h \leq s$.

I. Case I: $s < h \leq S$.--The beginning stock level can be any of the possible values from h to S inclusive. Therefore,

$$a_{r1}(h) = \sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq r \leq N \text{ and } s < h \leq S. \tag{10}$$

II. Case II: $h \leq s$.--The beginning stock level can be any of the possible values from s+1 to S inclusive. Therefore,

$$a_{r1}(h) = \sum_{i=s+1}^{S} a(i)P\{D(r-1)=i-h\} \quad \text{for } 1 \le r \le N \text{ and } h \le s. \quad (11)$$

III. Summary of $a_{r1}(h)$.--Expressions (10) and (11) are summarized as follows:

$$a_{r1}(h) = \begin{cases} \sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} & \text{for } 1 \le r \le N \text{ and } s < h \le S, \quad (12) \\ \sum_{i=s+1}^{S} a(i)P\{D(r-1)=i-h\} & \text{for } 1 \le r \le N \text{ and } h \le s. \end{cases}$$

Development of $a_{r2}(h)$.--The expression $a_{r2}(h)$ will be developed from the conditional stock level probabilities, $a_{r2}(h|x)$, corresponding to N mutually exclusive and exhaustive sets in conjunction with the probability of the specific replenishment lead time. That is,

$$a_{r2}(h) = \sum_{x=0}^{N-1} [a_{r2}(h|x)]P\{X=x\} \quad \text{for } 1 \le r \le N \text{ and } h \le S. \quad (13)$$

In this expression, $a_{r2}(h|x)$ is the probability that the stock level is equal to h prior to demand in period r and that the beginning stock level was equal to or less than s, given that the replenishment lead time is x periods.

I. Development of $a_{r2}(h|x)$.--The conditional probabilities that the stock level is equal to h prior to demand in period r and that the beginning stock level was equal to or less than s, given that the replenishment lead time is x periods, are denoted by $a_{r2}(h|x)$.

These conditional probabilities can be developed similarly to the development of the period stock level conditional stationary probabilities for the fixed cycle inventory policy under the hypothesis of back-orders allowed. The only difference is that S is replaced by s as an upper limit for h and the probability of $s < h \leq S$ is zero for $r < x+1$. The resulting expression is summarized as follows:

$$a_{r2}(h|x) = \begin{cases} \sum_{i=h}^{s} a(i)P\{D(r-1)=i-h\} & \text{for } 1 \leq r < x+1 \leq N \text{ and } h \leq s, \\ 0 & \text{for } 1 \leq r < x+1 \leq N \text{ and } s < h \leq S, \\ \delta_{hS} & \text{for } 1=r=x+1 \leq N, \\ P\{D(x)=S-h\} & \text{for } 1 < x+1 \leq r \leq N \text{ and } h \leq S. \end{cases} \qquad (14)$$

II. Unconditional $a_{r2}(h)$.--The unconditional probabilities that the period stock level is equal to h and that the beginning stock level was equal to or less than s, $a_{r2}(h)$, can be obtained by substituting expression (14) in expression (13).

Period stock level unconditional stationary probabilities.--The period stock level unconditional stationary probabilities can be obtained by inserting expressing (12) and (13) into expression (8).

The beginning stock level stationary probabilities and the period stock level stationary probabilities developed in this section will be used as the basis for determining the measures of effectiveness for the (s,S) inventory policy under the hypothesis that back-orders are allowed.

Back-Orders Not Allowed

The objective of this section is to develop expressions for the stock level probabilities under the hypothesis of back-orders not allowed. These expressions will be in terms of the demand and the replenishment lead time probability distributions; the order level, S; the lower order level, s; and the number of periods, N, in the order cycle.

When back-orders are not allowed, demand in excess of available stock is lost if not satisfied instantaneously by priority action. Under the (s,S) inventory policy a replenishment order is placed at the beginning of each order cycle if the beginning stock level, i, is equal to or less than s; otherwise, no replenishment order is placed. The replenishment quantity ordered depends upon the difference between the order level, S, and available stock. That is,

$$\text{replenishment order} = \begin{cases} 0 & \text{for } s < i \leq S, \\ S - \text{available stock} & \text{for } 0 \leq i \leq s. \end{cases} \tag{15}$$

Beginning Stock Level Stationary Probabilities

The probabilities that the beginning stock level is equal to i, a(i), after statistical equilibrium is attained, can be determined by solving a finite Markov transition matrix equation.

Markov transition matrix equation.--A transition from any beginning cycle stock level, i, to any stock level, j_2*, at the beginning of the second order cycle, for $0 \leq i \leq S$ and $0 \leq j_2 \leq S$, is possible as a consequence of the (s,S) ordering rule. Therefore, the transition matrix,

*The subscript on j_2 indicates the second consecutive order cycle.

$M = (m_{ij})$, is ergodic; and the stationary probabilities can be determined by solving the following matrix equation:

$$\underline{a} = \underline{a}\, M, \tag{16}$$

where $\underline{a} = [a(S),\ a(S-1),\ \ldots,\ a(s),\ a(s-1),\ \ldots,\ a(1),\ a(0)]$,

(16a)

$$\text{and } M = \begin{bmatrix} m_{S,S} & m_{S,S-1} & \cdots & m_{S,s} & \cdots & m_{S,0} \\ m_{S-1,S} & m_{S-1,S-1} & \cdots & m_{S-1,s} & \cdots & m_{S-1,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{s,S} & m_{s,S-1} & \cdots & m_{s,s} & \cdots & m_{s,0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{0,S} & m_{0,S-1} & \cdots & m_{0,s} & \cdots & m_{0,0} \end{bmatrix}.$$

In the finite matrix, M, m_{ij} is the probability that the stock level is j items at the beginning of the subsequent cycle given that the stock level was i items at the beginning of the present cycle.

Stock level transition probabilities.--In the development of the transition probabilities, consider the cases corresponding to the two mutually exclusive and exhaustive sets: $s < i \leq S$ and $0 \leq i \leq s$.

I. Case I: $s < i \leq S$.--No replenishment order is placed. Consider the sub-cases corresponding to the sub-sets obtained by trichotomizing this set into the sub-sets: $j=0$, $1 \leq j \leq i$, and $i < j \leq S$.

A. Sub-case I: $j=0$.--The stock level at the beginning of the subsequent cycle is equal to zero if the demand, $D(N)$, during the entire cycle is equal to or greater than i. Therefore,

$$m_{ij} = P\{D(N) \geq i\} \qquad \text{for } j=0 \text{ and } s < i \leq S. \tag{17}$$

B. <u>Sub-case II: $1 \leq j \leq i$</u>.--The stock level at the beginning of the subsequent cycle is equal to j if the demand, D(N), during the entire cycle is equal to i-j. Therefore,

$$m_{ij} = P\{D(N)=i-j\} \qquad \text{for } 1 \leq j \leq i \text{ and } s < i \leq S. \tag{18}$$

C. <u>Sub-case III: $i < j \leq S$</u>.--Since no replenishment order is placed, it is impossible that the beginning stock level, j, of the subsequent cycle is greater than the beginning stock level, i, of the present cycle. Therefore,

$$m_{ij} = 0 \qquad \text{for } i < j \leq S \text{ and } s < i \leq S. \tag{19}$$

D. <u>Summary of Case I</u>.--The unconditional transition probabilities, m_{ij}, for $s < i \leq S$ and $0 \leq j \leq S$, are summarized from expressions (17), (18), and (19) as follows:

$$m_{ij} = \begin{cases} P\{D(N) \geq i\} & \text{for } s < i \leq S \text{ and } j=0, \\ P\{D(N)=i-j\} & \text{for } s < i \leq S \text{ and } 1 \leq j \leq i \leq S, \\ 0 & \text{for } s < i \leq S \text{ and } i < j \leq S. \end{cases} \tag{20}$$

II. <u>Case II: $0 \leq i \leq s$</u>.--The stock level transition probabilities, m_{ij} for $0 \leq i \leq s$ and $0 \leq j \leq S$, can be developed similarly to the development of the stock level transition probabilities for the fixed cycle inventory policy under the hypothesis of back-orders not allowed, except that the upper bound for the stock level at the beginning of the present cycle is s rather than S. The resulting expression is summarized as follows:

$$m_{ij} = \begin{cases} P\{D(N) \geq S\}P\{X=0\} + \sum_{x=1}^{N-1} \Big[P\{D(x) \geq i\}P\{D(N-x) \geq S-i\} \\ \quad + \sum_{h=1}^{i} P\{D(x) \geq i-h\}P\{D(N-x)=h+S-i\} \Big] P\{X=x\} \\ \qquad \text{for } 0 \leq i \leq s \text{ and } j=0, \\ P\{D(N)=S-j\}P\{X=0\} + \sum_{x=1}^{N-1} \Big[P\{D(x) \geq i\}P\{D(N-x)=S-i-j\} \\ \quad + \sum_{h=1}^{i} P\{D(x)=i-h\}P\{D(N-x)=h+S-i-j\} \Big] P\{X=x\} \\ \qquad \text{for } 0 \leq i \leq s \text{ and } 1 \leq j \leq S. \end{cases} \tag{21}$$

The stationary probabilities for the stock level at the beginning of the order cycle can be determined from expression (16) utilizing the transition probabilities from expressions (20) and (21).

Period Stock Level Stationary Probabilities

The probabilities that the stock level is equal to h prior to demand in period r are denoted by $a_r(h)$. These probabilities will be determined by combining the probabilities of two mutually exclusive and exhaustive sets. That is,

$$\begin{aligned} &P\{\text{stock level}=h \text{ prior to demand in period } r\} \\ &= P\{\text{stock level}=h \text{ prior to demand in period } r,\ s < i \leq S\} \\ &\quad + P\{\text{stock level}=h \text{ prior to demand in period } r,\ 0 \leq i \leq s\}, \\ &\qquad \text{for } 1 \leq r \leq N \text{ and } 0 \leq h \leq S. \end{aligned} \tag{22}$$

The probabilities that the stock level prior to the demand during period r is equal to h and that the inventory at the beginning of the cycle is greater than s, but not in excess of S, are denoted by $a_{r1}(h)$. The probabilities that the stock level prior to the demand during period r is equal to h and that the inventory at the beginning of the cycle is equal to or less than s, but non-negative, are denoted by $a_{r2}(h)$. Therefore,

$$a_r(h) = a_{r1}(h) + a_{r2}(h) \qquad \text{for } 1 \leq r \leq N \text{ and } 0 \leq h \leq S. \tag{23}$$

Development of $a_{r1}(h)$.--Consider the cases corresponding to the three mutually exclusive and exhaustive sets: $s < h \leq S$, $1 \leq h \leq s$, and $h=0$.

I. Case I: $s < h \leq S$.--A specific path in which the stock level is equal to h prior to the demand in period r occurs if the beginning stock level is equal to i, for $s < i \leq S$, and if the demand, $D(r-1)$, during the r-1 periods is equal to i-h. The probability of this specific path is

$$a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq r \leq N,\ s < i \leq S, \text{ and } 0 \leq h \leq i. \tag{24}$$

The beginning stock level can be any of the possible values from h to S inclusive. Therefore,

$$a_{r1}(h) = \sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} \qquad \text{for } 1 \leq r \leq N \text{ and } s < h \leq S. \tag{25}$$

II. Case II: $1 \leq h \leq s$.--This case is similar to Case I except that the beginning stock level can be any of the possible values from s+1 to S inclusive. Therefore,

$$a_{r1}(h) = \sum_{h=s+1}^{S} a(i)P\{D(r-1)=i-h\} \quad \text{for } 1 \leq r \leq N \text{ and } 1 \leq h \leq s. \tag{26}$$

III. Case III: h=0.--A specific path in which the stock level is equal to zero prior to the demand in period r occurs if the demand, D(r-1), during the r-1 periods is equal to or greater than the beginning cycle stock level, i. The beginning stock level can be any of the possible values from s+1 to S inclusive. Therefore,

$$a_{r1}(h) = \sum_{i=s+1}^{S} a(i)P\{D(r-1) \geq i\} \quad \text{for } 1 \leq r \leq N \text{ and } h=0. \tag{27}$$

IV. Summary of $a_{r1}(h)$.--Expressions (25), (26) and (27) are summarized as follows:

$$a_{r1}(h) = \begin{cases} \sum_{i=h}^{S} a(i)P\{D(r-1)=i-h\} \text{ for } 1 \leq r \leq N \text{ and } s < h \leq S, \\ \sum_{i=s+1}^{S} a(i)P\{D(r-1)=i-h\} \text{ for } 1 \leq r \leq N \text{ and } 1 \leq h \leq s, \\ \sum_{i=s+1}^{S} a(i)P\{D(r-1) \geq i\} \text{ for } 1 \leq r \leq N \text{ and } h=0. \end{cases} \tag{28}$$

Development of $a_{r2}(h)$.--The expression, $a_{r2}(h)$, will be developed from the conditional stock level probabilities, $a_{r2}(h|x)$, corresponding to N mutually exclusive and exhaustive sets, in conjunction with the probability of the specific replenishment lead time. That is,

$$a_{r2}(h) = \sum_{x=0}^{N-1} [a_{r2}(h|x)]P\{X=x\} \qquad \text{for } 1 \leq r \leq N \text{ and } 0 \leq h \leq S. \tag{29}$$

In this expression, $a_{r2}(h|x)$ is the probability that the stock level is equal to h prior to the demand in period r and that the beginning stock level is equal to or less than s, given that the replenishment lead time is x periods.

I. Development of $a_{r2}(h|x)$.--The conditional stock level probabilities that the stock level is equal to h prior to the demand in period r and that $0 \leq i \leq s$, given that the replenishment lead time is x periods, are denoted by $a_{r2}(h|x)$. These probabilities will be developed in terms of the beginning stock level stationary probabilities, $a(i)$, from the relationship of the period, r, within the order cycle to the replenishment lead time, X=x. In this development, consider the cases corresponding to the three mutually exclusive and exhaustive sets:

A. Case I: $r < x+1$.--The stock level conditional probabilities, $a_{r2}(h|x)$, will be developed by considering the sub-cases corresponding to the sub-sets obtained by trichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $h=0$, $1 \leq h \leq s$, and $s < h \leq S$.

1. Sub-case I: $h=0$.--A specific path in which the stock level prior to the demand in period r is equal to zero occurs if the beginning stock level is equal to i and if the demand, $D(r-1)$, during the r-1 periods is equal to or greater than i. Therefore, the probability of this specific path is

$$a(i)P\{D(r-1) \geq i\} \quad \text{for } 1 \leq r < x+1 \leq N,\ 0 \leq i \leq s, \text{ and } h=0. \tag{30}$$

Figure 13 (page 55) illustrates a similar path for the fixed cycle inventory policy.

The beginning stock level can be any of the possible values from 0 to s inclusive. Therefore,

$$a_{r2}(h|x) = \sum_{i=0}^{s} a(i)P\{D(r-1) \geq i\} \quad \text{for } 1 \leq r < x+1 \leq N \text{ and } h=0. \tag{31}$$

2. Sub-case II: $1 \leq h \leq s$.--A specific path in which the stock level prior to the demand in period r is equal to h occurs if the beginning stock level is equal to i, for $1 \leq h \leq i \leq s$, and if the demand, $D(r-1)$, during the r-1 periods is equal to i-h. Therefore, the probability of this specific path is

$$a(i)P\{D(r-1)=i-h\} \quad \text{for } 1 \leq h \leq i \leq s \text{ and } 1 \leq r < x+1 \leq N. \tag{32}$$

Figure 14 (page 55) illustrates a similar path for the fixed cycle inventory policy.

The beginning stock level can be any of the possible values from h to s inclusive. Therefore,

$$a_{r2}(h|x) = \sum_{i=h}^{s} a(i)P\{D(r-1)=i-h\} \quad \text{for } 1 \leq h \leq s \text{ and } 1 \leq r < x+1 \leq N. \tag{33}$$

3. <u>Sub-case III: $s < h \leq S$</u>.--Since the beginning stock level is no greater than s and since the replenishment order has not been received, it is impossible that the stock level at the beginning of period r is greater than s. Therefore,

$$a_{r2}(h|x) = 0 \quad \text{for } 1 \leq r < x+1 \leq N,\ 0 \leq i \leq s, \text{ and } s < h \leq S. \tag{34}$$

B. <u>Case II: r=x+1</u>.--The stock level conditional probabilities for this case will be developed by considering the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: x=0 and $1 \leq x \leq N-1$.

1. <u>Sub-case I: x=0</u>.--If the replenishment order is received in zero time periods, the stock level prior to the demand in the first period is instantaneously S items. Therefore,

$$a_{1,2}(h|0) = \delta_{hS}. \tag{35}$$

2. <u>Sub-case II: $1 \leq x \leq N-1$</u>.--When the replenishment lead time is positive, there is the possibility that the demand during the replenishment lead time is greater than the quantity of items in the beginning stock. Consider the sub-cases corresponding to the sub-sets

obtained by dichotomizing this case into the following mutually exclusive and exhaustive sub-sets: $S \geq h \geq S-s$ and $S-s > h \geq 0$.

a. Sub-case IIA: $S \geq h \geq S-s$.--The conditional cumulative probability distribution will be useful in obtaining an expression for this sub-case. The stock level at the beginning of period r, for $r=x+1$, is equal to or less than some specified value, say h, for $S \geq h \geq S-s$, if (1) the stock level at the beginning of the order cycle is equal to or greater than S-h items, but not greater than s items; and if (2) the demand during the replenishment lead time, D(x), is equal to or greater than S-h items. As a consequence of the beginning stock level, a replenishment order is placed for no more than h items, but for at least S-s items.

Algebraically, the previous argument is as follows. If the beginning stock level is i where $S-h \leq i \leq s$, then S-i items are ordered. If demand exists during the replenishment lead time such that $D(x) \geq S-h$, then the stock level available prior to the receipt of the replenishment order is max $(0, i-D(x))$. If $D(x) \geq S-h$, there are at most $i-(S-h)$ items available when the order is received. By hypothesis $i-(S-h)$ is non-negative. Therefore, if Y denotes the number of items available after the replenishment order is received, then

$$Y \leq [i-(S-h)] + (S-i) \quad \text{for } S-h \leq i \leq s \text{ and } s \geq h \geq S-s, \tag{36}$$

$$\text{or } Y \leq h \quad \text{for } S \geq h \geq S-s. \tag{37}$$

The following cumulative probability functions are defined:

$$A(S-h-1) = \sum_{i=0}^{S-h-1} a(i) \quad \text{and} \quad A_{r2}(h|x) = \sum_{i=0}^{h} a_{r2}(i|x) \tag{38}$$

for $1 \leq r = x+1 \leq N$ and $0 \leq h \leq S$.

From the above argument,

$$A_{x+1,2}(h|x) = [A(s) - A(S-h-1)]P\{D(x) \geq S-h\} \tag{39}$$

for $1 < x+1 \leq N$ and $S \geq h \geq S-s$.

By the definition of a discrete probability,

$$a_{x+1,2}(h|x) = A_{x+1,2}(h|x) - A_{x+1,2}(h-1|x) \tag{40}$$

for $1 < r=x+1 \leq N$ and $S \geq h \geq S-s$,

where $A_{r2}(S-s-1|x) = 0$ by definition.

Therefore,

$$\begin{aligned} a_{x+1,2}(h|x) = {} & [A(s) - A(S-h-1)]P\{D(x) \geq S-h\} \\ & -[A(s) - A(S-h)]P\{D(x) \geq S-h+1\} \end{aligned} \tag{41}$$

for $1 < x+1 \leq N$ and $S \geq h \geq S-s$.

b. Sub-case IIB: $S-s > h \geq 0$.--Since the development of $a_{r2}(h(x)$ imples that the beginning stock level is equal to ore less than s, there are at the minimum S-s items in the replenishment order. Therefore, the quantity of items available when the replenishment order is received is no smaller than the minimum replenishment order, S-s. That is,

$$a_{r2}(h|x) = 0 \qquad \text{for } 1 < r=x+1 \leq N \text{ and } 0 \leq h < S-s. \tag{42}$$

C. Case III: $r > x+1$.--The period stock level conditional probabilities for this case will be obtained in terms of the stock level probabilities when the replenishment order is received, $a_{x+1}(h|x)$. Consider the sub-cases corresponding to the mutually exclusive and exhaustive sub-sets obtained by trichotomizing this set into $h=0$, $1 \leq h < S-s$, and $S-s \leq h \leq S$.

1. Sub-case I: $h=0$.--A specific path in which the stock level at the beginning of period r is equal to zero occurs if the stock level after the replenishment order is received is ν and if the demand, $D(r-x-1)$, during the r - (x+1) periods is equal to or greater than ν. Therefore, the probability of this specific path is

$$[a_{x+1,2}(\nu|x)]P\{D(r-x-1) \geq \nu\} \quad \text{for } 1 < x+1 < r \leq N \text{ and } S-s \leq \nu \leq S. \tag{43}$$

Figure 15 (page) illustrates a similar path for the fixed cycle inventory policy.

The range of ν begins with S-s since this quantity is the minimum order quantity with the (s,S) inventory policy. The stock level when the replenishment order is received can be any of the possible values from S-s to S inclusive. Therefore,

$$a_{r2}(h|x) = \sum_{\nu=S-s}^{S} [a_{x+1,2}(\nu|x)]P\{D(r-x-1 \geq \nu\} \tag{44}$$

$$\text{for } 1 \leq x+1 < r \leq N \text{ and } h=0.$$

2. Sub-case II: $1 \leq h < S-s$.--A specific path in which the stock level prior to the demand in period r is equal to h occurs if the

stock level after the replenishment order is received is $h+\nu$ and if the demand, $D(r-x-1)$, during the $r - (x+1)$ periods is ν. Therefore, the probability of this specific path is

$$[a_{x+1,2}(h+\nu|x)]P\{D(r-x-1)=\nu\} \qquad \text{for } 1 \leq x+1 < r \leq N, \tag{45}$$
$$\nu \geq 0, \ \ S-s \leq h+\nu \leq \nu \leq S, \text{ and } 1 \leq h < S-s.$$

Figure 16 (page 55) illustrates a similar path for the fixed cycle inventory policy.

Possible values of the stock level $(h+\nu)$ when the replenishment order is received are from $S-s$ to S inclusive. Therefore,

$$a_{r2}(h|x) = \sum_{\nu=S-s-h}^{S-h} [a_{x+1,2}(h+\nu|x)]P\{D(r-x-1)=\nu\} \tag{46}$$
$$\text{for } 1 \leq x+1 < r \leq N \text{ and } 1 \leq h < S-s.$$

3. Sub-case III: $S-s \leq h \leq S$.--This sub-case is similar to Sub-case II. However, possible values for the stock level $(h+\nu)$ when the replenishment order is received are from h to S inclusive. Therefore,

$$a_{r2}(h|x) = \sum_{\nu=0}^{S-h} [a_{x+1,2}(h+\nu|x)]P\{D(r-x-1)=\nu\} \tag{47}$$
$$\text{for } 1 \leq x+1 < r \leq N \text{ and } S-s \leq h \leq S.$$

D. Summary of $a_{r2}(h|x)$.--Expressions (31), (33), (34), (35), (41), (42), (44), (46), and (47) are summarized as follows:

$$
a_{r2}(h|x) = \begin{cases}
\displaystyle\sum_{i=0}^{s} a(i)P\{D(r-1) \geq i\} & \text{for } 1 \leq r < x+1 \leq N \text{ and } h=0. \\[2ex]
\displaystyle\sum_{i=h}^{s} a(i)P\{D(r-1)=i-h\} & \text{for } 1 \leq r < x+1 \leq N \text{ and } 1 \leq h \leq s, \\[2ex]
0 & \text{for } 1 \leq r < x+1 \leq N \text{ and } s < h \leq S, \\[2ex]
\delta_{hS} & \text{for } 1 = r = x+1 \leq N, \\[2ex]
[A(s)-A(S-h-1)]P\{D(x) \geq S-h\} & \\
\quad - [A(s)-A(S-h)]P\{D(x) \geq S-h+1\} & \text{for } 1 < r = x+1 \leq N \text{ and } S-s \leq h \leq S, \\[2ex]
0 & \text{for } 1 < r = x+1 \leq N \text{ and } 0 \leq h < S-s, \\[2ex]
\displaystyle\sum_{\nu=S-s}^{S} [a_{x+1,2}(\nu|x)]P\{D(r-x-1) \geq \nu\} & \text{for } 1 \leq x+1 < r \leq N \text{ and } h=0, \\[2ex]
\displaystyle\sum_{\nu=S-s-h}^{S-h} [a_{x+1,2}(h+\nu|x)]P\{D(r-x-1) \geq \nu\} & \text{for } 1 \leq x+1 < r \leq N \text{ and } 1 \leq h < S-s, \\[2ex]
\displaystyle\sum_{\nu=0}^{S-h} [a_{x+1,2}(h+\nu|x)]P\{D(r-x-1)=\nu\} & \text{for } 1 \leq x+1 < r \leq N \text{ and } S-s \leq h \leq S.
\end{cases} \tag{48}
$$

II. Unconditional $a_{r2}(h)$.--The unconditional probabilities that the period stock level is equal to h and that a replenishment order is placed, $a_{r2}(h)$, can be obtained by substituting expression (48) into expression (29).

Period stock level unconditional stationary probabilities.--The period stock level stationary probabilities developed in this section will be used as the basis for determining the measures of effectiveness for the (s,S) inventory policy under the hypothesis that back-orders are not allowed.

Results

The expression for the stock level probabilities for the (s,S) inventory policy have been developed under the hypothesis of back-orders allowed and the hypothesis of back-orders not allowed. The results of this chapter will be used in Chapter VII in the determination of the measures of effectiveness required to attain the primary objective of the study.

CHAPTER V

LENGTH-OF-CYCLE PROBABILITIES AND STOCK LEVEL PROBABILITIES FOR THE VARIABLE CYCLE INVENTORY POLICY

Introduction

The objective of this chapter is to develop mathematical expressions for the length-of-cycle probabilities and the stock level probabilities in terms of relevant controlled decision variables characteristic of the variable cycle inventory policy and uncontrolled random variables. The variable cycle inventory policy requires that a replenishment order be placed at the beginning of the first period in which the stock level is equal to or below the reorder point. This order is for a quantity equal to the difference between the order level, S, and the beginning stock level, i. The relevant controlled decision variables are the order level, S, and the reorder point, RP. The uncontrolled random variables are the demand and replenishment lead time. That the number of periods between successive replenishment orders is variable follows from the random features of the demand and of the replenishment lead time. These stock level probabilities will be developed separately for the hypothesis of back-orders allowed and for the hypothesis of back-orders not allowed.

This inventory policy is similar to a policy often designated as the fixed quantity inventory policy. Under the fixed quantity inventory policy, it is assumed that stock level surveillance is continuous, that a replenishment order can be placed at any time with the vendor, and that each demand is for only one item. The treatment of the variable cycle inventory policy in the present study will eliminate these usual assumptions. The assumptions listed and discussed in Chapter I will be used as a basis for the analytical development of the variable cycle inventory policy considered in this chapter. The replenishment lead time is assumed to be equal to or less than K periods. Also, the demand during any period is assumed to be less than (S-RP)/K with probability equal to one. Therefore, no replenishment order is placed while the previous order is outstanding.

Length-of-Cycle Probabilities, Back-Orders-Allowed

The objective of this section is to develop expressions for the length-of-cycle probabilities under the hypothesis of back-orders allowed. These probabilities will be developed in terms of the demand probability distribution; the order level, S; and the reorder point, RP.

When back-orders are allowed, unsatisfied demands are deferred until sufficient stock is available. Under the variable cycle inventory policy a replenishment order is placed at the end of the first period in which the stock level is equal to or below the reorder point, RP. The replenishment quantity ordered depends either upon (1) the difference between order level, S, and available stock or upon (2) the sum of unsatisfied demand and order level, S. Therefore,

$$\text{replenishment order} = \begin{cases} S - \text{available stock} & \text{for } i > 0, \\ S + \text{unsatisfied demand} & \text{for } i \leq 0. \end{cases} \qquad (1)$$

The random variable, Ω, denotes the length of the variable order cycle, and ω = K+1, K+2, . . . are the values which this random variable assumes. The probability that the random variable, Ω, is equal to ω periods is expressed by $P\{\Omega=\omega\}$.

A specific path in which the number of periods within the variable length order cycle is equal to ω occurs if demand during the first ω-1 periods is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1$, and if demand during the next period is equal to or greater than S-RP-ν. The probability of this specific path is

$$P\{D(\omega-1)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}\nu\} \qquad \text{for } K < \omega \text{ and } 0 \leq \nu \leq S\text{-}RP\text{-}1. \tag{2}$$

Demand during the first ω-1 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore,

$$P\{\Omega=\omega\} = \sum_{\nu=0}^{S\text{-}RP\text{-}1} P\{D(\omega-1)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}\nu\} \text{ for } K < \omega. \tag{3}$$

The above expression for the length-of-cycle probabilities will be used (1) in the development of the stock level probabilities in the subsequent section and (2) in the determination of the measures of effectiveness for the variable cycle inventory policy under the hypothesis of back-orders allowed.

Stock Level Probabilities, Back-Orders Allowed

The objective of this section is to develop expressions for the stock level probabilities under the hypothesis of back-orders allowed.

These probabilities will be developed in terms of the demand and the replenishment lead time probability distributions; the order level, S; and the reorder point, RP. The beginning stock level probabilities will be unconditional; however, the period stock level probabilities will be developed upon the condition of a specific length of cycle.

Beginning Stock Level Stationary Probabilities

The probabilities that the beginning stock level is equal to i, a(i), after statistical equilibrium is attained, will be determined by solving an infinite Markov transition matrix equation.

Markov transition matrix equation.--A transition from any beginning cycle stock level, i, to any stock level, j, at the beginning of the subsequent variable length cycle, for $i \leq RP$ and $j \leq RP$, is possible as a consequence of the ordering rule. Therefore, the transition matrix, $M = (m_{ij})$, is regular; and the stationary probabilities exist and are unique. These stationary probabilities will be determined by solving the following matrix equation:

$$\underline{a} = \underline{a}\, M, \tag{4}$$

$$\text{where } \underline{a} = [a(RP),\ a(RP-1),\ .\ .\ .,\ a(0),\ .\ .\ .], \tag{4a}$$

$$\text{and } M = \begin{bmatrix} m_{RP,RP} & m_{RP,RP-1} & \cdots & m_{RP,0} & \cdots \\ m_{RP-1,RP} & m_{RP-1,RP-1} & \cdots & m_{RP-1,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{0,RP} & m_{0,RP-1} & \cdots & m_{0,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{4b}$$

In the infinite matrix, M, m_{ij} is the probability that the stock level is j items at the beginning of the subsequent variable length cycle, given that the stock level was i items at the beginning of the present variable length cycle.

Stock level transition probabilities.--The stock level transition probability, m_{ij}, will be obtained from the joint stock level transition probability elements, $m_{ij,\omega}$, for $\omega = K+1, K+2, \ldots$. The joint stock level transition element, $m_{ij,\omega}$, is the probability that the stock level at the beginning of the subsequent cycle is j items and that the number of periods in the variable length cycle is ω, given that the stock level at the beginning of the present variable length cycle is i items. The random variable, Ω, corresponds to the number of periods in the variable length cycle and $\omega = K+1, K+2, \ldots$ are the values which this random variable assumes. The events $\Omega = \omega$ are mutually exclusive and exhaustive, and the joint stock level transition probabilities can be added. Therefore,

$$m_{ij} = \sum_{\omega=K+1}^{\infty} m_{ij,\omega} \qquad \text{for i and j} \leq \text{RP}. \tag{5}$$

A specific path in which the stock level at the beginning of the subsequent cycle is equal to j and in which the number of periods in the variable length cycle is equal to ω occurs if demand during the first ω-1 periods is equal to ν, for $0 \leq \nu \leq$ S-RP-1, and if demand during the next period is equal to S-ν-j. The probability of this specific path is

$$P\{D(\omega-1)=\nu\}P\{D(1)=S-\nu-j\} \qquad \text{for } K < \omega,\ 0 \leq \nu \leq S\text{-}RP\text{-}1, \qquad (6)$$

$$\text{and } j \leq RP.$$

Demand during the first ω-1 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore,

$$m_{ij,\omega} = \sum_{\nu=0}^{S-RP-1} P\{D(\omega-1)=\nu\}P\{D(1)=S-\nu-j\} \qquad (7)$$

$$\text{for } K < \omega,\ i \leq RP, \text{ and } j \leq RP.$$

The stock level transition probabilities, m_{ij}, will be obtained by inserting expression (7) into expression (5).

The stationary probabilities that the beginning stock level is equal to i, a(i), will be determined from expression (4). Since the rows of the matrix are independent of i (identical), the stationary probabilities are obtained by inserting expression (7) into expression (5). That is,

$$a(j) = \sum_{\omega=K+1}^{\infty} m_{ij,\omega} \qquad \text{for } i \leq RP \text{ and } j \leq RP. \qquad (8)$$

By a change in subscript notation,

$$a(i) = \sum_{\omega=K+1}^{\infty} m_{ji,\omega} \qquad \text{for } i \leq RP \text{ and } j \leq RP. \qquad (9)$$

Period Stock Level Conditional Stationary Probabilities

The conditional stationary probabilities that the stock level is equal to h prior to the demand in period r, given ω periods in the cycle, are denoted by $a_r(h|\omega)$. These probabilities will be developed from the period stock level conditional probabilities, $a_r(h|\omega,x)$ corresponding to K+1 mutually exclusive and exhaustive sets, in conjunction with the conditional probability of the lead time for a given length of cycle. That is,

$$a_r(h|\omega) = \sum_{x=0}^{K} a_r(h|\omega,x)P\{X=x|\omega\} \tag{10}$$

$$\text{for } 1 \leq r \leq \omega,\ K < \omega,\ \text{and } h \leq RP.$$

Under the hypothesis of back-orders allowed, X and Ω are independent. Therefore, expression (10) is reduced to the following:

$$a_r(h|\omega) = \sum_{x=0}^{K} a_r(h|\omega,x)P\{X=x\} \tag{11}$$

$$\text{for } 1 \leq r \leq \omega,\ K < \omega,\ \text{and } h \leq RP.$$

Period stock level double conditional stationary probabilities.--The double conditional probabilities that the stock level is equal to h prior to the demand in period r, given that ω periods are in the order cycle and given that the replenishment lead time is equal to x periods for the beginning-cycle order, are denoted by $a_r(h|\omega,x)$. These probabilities will be developed from the following expression:

$$a_r(h|\omega,x) = \frac{P\{H_r=h,\ \Omega=\omega|X=x\}}{P\{\Omega=\omega\ |X=x\}}, \tag{12}$$

where H_r is the stock level random variable at period r and h=S, S-1, . . . are the values which this random variable may assume. Since the random variables X and Ω are independent, the denominator reduces to the length-of-cycle probabilities previously developed in expression (3). In this development of the numerator of expression (12), consider the mutually exclusive and exhaustive sets: $r < x+1$ and $r \geq x+1$.

I. Case I: $r < x+1$.--The joint probabilities that the stock level is equal to h prior to the demand in period r and that the length of cycle is ω periods, given that the lead time is equal to x, will be developed. Consider the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $h \leq RP$ and $RP < h \leq S$.

A. Sub-case I: $h \leq RP$.--A specific path in which the stock level is equal to h, for $h \leq RP$, prior to the demand in period r, for $1 \leq r < x+1 \leq K+1$, and in which the length-of-cycle is equal to ω occurs if the following four independent conditions exist:

1. The beginning stock level is equal to i.
2. Demand during the first r-1 periods is equal to i-h.
3. Demand during the next ω-r periods is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}i\text{+}h\text{-}1$.
4. Demand during period ω is equal to or greater than $S\text{-}RP\text{-}i\text{+}h\text{-}\nu$.

For specific values of i and ν, the probability of this path is

$$a(i)P\{D(r-1)=i-h\}P\{D(\omega-r)=\nu\}P\{D(1) \geq S-RP-i+h-\nu\} \tag{13}$$

for $h \leq i \leq RP$, $1 \leq r < x+1 \leq K+1$, and $0 \leq \nu \leq S-RP-i+h-1$.

The beginning stock level can be any of the possible values from h to RP inclusive. Demand during the ω-r periods can be any of the possible values from 0 to S-RP-i+h-1 inclusive. Therefore,

$$P\{H_r=h,\ \Omega=\omega \mid X=x\} = \tag{14}$$

$$\sum_{i=h}^{RP} \sum_{\nu=0}^{S-RP-i-1} G(i,\nu)P\{D(1) \geq S-RP-i+h-\nu\},$$

$$\text{where } G(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-r)=\nu\}, \tag{14a}$$

$$\text{for } 1 \leq r < x+1 \leq K+1 \text{ and } h \leq RP.$$

B. Sub-case II: RP < h ≤ S.--It is impossible that the stock level at the beginning of period r, for $1 \leq r < x+1 \leq K+1$, is greater than RP. Therefore,

$$P\{H_r=h,\ \Omega=\omega \mid X=x\} = 0 \tag{15}$$

$$\text{for } 1 \leq r < x+1 \leq K+1 \leq \omega \text{ and } RP < h \leq S.$$

II. Case II: r ≥ x+1.--The joint probabilities that the stock level is equal to h prior to the demand in period r and that the length of cycle is ω periods, given that the lead time is equal to x, will be developed. Consider the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $h \leq RP$ and $RP < h \leq S$.

A. Sub-case I: h ≤ RP.--If the stock level is equal to or less than RP, a replenishment order has been placed. Since the placement of a replenishment order constitutes the beginning of an order cycle, the period is considered as being in the subsequent variable order cycle. Therefore,

$$P\{H_r=h,\ \Omega=\omega \mid X=x\} = 0 \tag{16}$$

for $1 \leq x+1 \leq r \leq \omega$ and $h \leq RP$.

B. Sub-case II: $RP < h \leq S$.--A specific path in which the stock level is equal to h, for $RP < h \leq S$, prior to the demand in period r, for $1 \leq x+1 \leq r \leq \omega$, and in which the length of cycle is equal to ω occurs if the following three independent conditions exist:

1. Demand during the first r-1 periods is equal to S-h.
2. Demand during the next ω-r periods is equal to ν, for $0 \leq \nu \leq h-RP-1$.
3. Demand during period ω is equal to or greater than h-RP-ν.

For a specific value of ν, the probability of this path is

$$P\{D(r-1)=S-h\}P\{D(\omega-r)=\nu\}P\{D(1) \geq h-RP-\nu\} \tag{17}$$

for $1 \leq x+1 \leq r \leq \omega$, $0 \leq \nu \leq h-RP-1$, and $RP < h \leq S$.

Demand during the ω-r periods can be any of the possible values from 0 to h-RP-1 inclusive. Therefore,

$$P\{H_r=h,\ \Omega=\omega \mid X=x\} = \sum_{\nu=0}^{h-RP-1} G(\nu)P\{D(1) \geq h-RP-\nu\}, \tag{18}$$

$$\text{where } G(\nu) = P\{D(r-1)=S-h\}P\{D(\omega-r)=\nu\}, \tag{18a}$$

for $1 \leq x+1 \leq r \leq \omega$ and $RP < h \leq S$.

III. Summary of period stock level double conditional stationary probabilities.--Expressions (14), (15), (16), and (18) are inserted into expression (12) and are summarized as follows:

$$a_r(h|\omega,x) = \begin{cases} \dfrac{\displaystyle\sum_{i=h}^{RP} \sum_{\nu=0}^{S-RP-i-1} G(i,\nu)P\{D(1) \geq S\text{-}RP\text{-}i\text{+}h\text{-}\nu\}}{P\{\Omega=\omega\}} & \\ \qquad \text{for } 1 \leq r < x+1 \leq K+1 \text{ and } h \leq RP, & \\ 0 \qquad \text{for } 1 \leq r < x+1 \leq K+1 \text{ and } RP < h \leq S, & \\ 0 \qquad \text{for } 1 \leq x+1 \leq r \leq \omega \text{ and } h \leq RP, & \\ \dfrac{\displaystyle\sum_{\nu=0}^{h-RP-1} G(\nu)P\{D(1) \geq h\text{-}RP\text{-}\ \}}{P\{\Omega=\omega\}} & \\ \qquad \text{for } 1 \leq x+1 \leq r \leq \omega \text{ and } RP < h \leq S, & \end{cases} \tag{19}$$

$$\text{where } G(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-r)=\nu\} \tag{19a}$$

$$\text{and } G(\nu) = P\{D(r-1)=S-h\}P\{D(\omega-r)=\nu\}. \tag{19b}$$

Period stock level conditional stationary probabilities.--The period stock level conditional probabilities, $a_r(h|\omega)$, can be obtained by inserting expression (19) into expression (11).

The beginning stock level stationary probabilities and the period stock level conditional probabilities developed in this section will be used as the basis for determining the measures of effectiveness for the variable cycle inventory policy under the hypothesis that backorders are allowed.

Length-of-Cycle Probabilities, Back-Orders Not Allowed

The objective of this section is to develop expressions for the length-of-cycle probabilities under the hypothesis of back-orders not allowed. These probabilities will be developed in terms of the demand and the replenishment lead time probability distributions; the order level, S; and the reorder point, RP. The random variable, Ω, denotes the length of the variable cycle, and ω = K+1, K+2, ... are the values which this random variable assumes. The probability that the random variable, Ω, is equal to ω periods is denoted by $P\{\Omega=\omega\}$.

When back-orders are not allowed, demand that is in excess of available stock is lost if not satisfied instantaneously by priority action. Under the variable cycle inventory policy a replenishment order is placed at the end of the first period in which the stock level is equal to or below the reorder point, RP. The replenishment quantity ordered depends upon only the difference between the order level, S, and the available stock, i. Therefore,

$$\text{replenishment order} = S-i \qquad \text{for } 0 \leq i \leq RP. \tag{20}$$

Since demands which occur when stock is unavailable do not result in a reduction of stock level, the average number of periods per cycle under the hypothesis that back-orders are not allowed is equal to or greater than under the hypothesis that back-orders are allowed.

The length-of-cycle probabilities, $P\{\Omega=\omega\}$, will be developed from the length-of-cycle conditional probabilities, $P\{\Omega=\omega|x\}$, corresponding to K+1 mutually exclusive and exhaustive sets, in conjunction with the probability of the specific replenishment lead time. That is,

$$P\{\Omega=\omega\} = \sum_{x=0}^{K} P\{\Omega=\omega|x\}P\{X=x\} \qquad \text{for } K < \omega. \tag{21}$$

Length-of-Cycle Conditional Probabilities

In developing the length-of-cycle conditional probabilities, given that the replenishment lead time is equal to x periods, consider the paths corresponding to the mutually exclusive and exhaustive sets in which $D(x) \geq i$ and $D(x) < i$.

Path I: $D(x) \geq i$.--When demand during the replenishment lead time is equal to or greater than the beginning stock level, i, the quantity of items available prior to the receipt of the replenishment order is zero. Since the replenishment order is for S-i items, the quantity of items available when the replenishment order is received is S-i. A specific path in which the number of periods in the order cycle is equal to ω occurs if the following four independent conditions exist:

1. The beginning stock level is equal to i, for $0 \leq i \leq RP$.
2. The demand during the lead time is equal to or greater than i.
3. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq$ S-RP-1-i.
4. Demand during period ω is equal to or greater than S-RP-ν-i.

For specific values of i and ν, the probability of this path is

$$a(i)P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\}P\{D(1) \geq S-RP-\nu-i\} \tag{22}$$

for $0 \leq i \leq RP$, $1 \leq x+1 \leq K+1 \leq \omega$, and $0 \leq \nu \leq$ S-RP-1-i.

The beginning stock level can be any of the possible values from 0 to RP inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1-i inclusive. Therefore, the probability of Path I is

$$\sum_{i=0}^{RP} \sum_{\nu=0}^{S-RP-1-i} G(i,\nu)P\{D(1) \geq S-RP-\nu-i\}, \tag{23}$$

$$\text{where } G(i,\nu) = a(i)P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\}, \tag{23a}$$

$$\text{for } 1 \leq x+1 \leq K+1 \leq \omega.$$

Path II: D(x) < i.--When demand during the replenishment lead time is less than i, denote the quantity of items available prior to the receipt of the replenishment order as h. The quantity of items available after the replenishment order is received is h+S-i. A specific path in which the number of periods in the order cycle is equal to ω occurs if the following four independent conditions exist:

1. The beginning stock level is equal to i, for $0 \leq i \leq RP$.
2. Demand during the lead time is equal to i-h, which is less than i.
3. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S-RP-1-i+h$.
4. Demand during period ω is equal to or greater than S-RP-ν-i+h.

For specific values of i, h, and ν, the probability of this path is

$$a(i)P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}P\{D(1) \geq S-RP-\nu-i+h\} \tag{24}$$

$$\text{for } 0 \leq i \leq RP,\ 1 \leq x+1 \leq K+1 \leq \omega, \text{ and } 0 \leq \nu \leq S-RP-1-i+h.$$

The beginning stock level can be any of the possible values from 0 to RP inclusive. Demand during the replenishment lead time can be any of the possible values from 1 to i inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1-i+h inclusive. Therefore, the probability of Path II is

$$\sum_{i=0}^{RP} \sum_{h=1}^{i} \sum_{\nu=0}^{S-RP-1-i+h} G(i,h,\nu)P\{D(1) \geq S\text{-}RP\text{-}\nu\text{-}i\text{+}h\}, \tag{25}$$

$$\text{where } G(i,h,\nu) = a(i)P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{25a}$$

$$\text{for } 1 \leq x+1 \leq K+1 \leq \omega.$$

Summary of the length-of-cycle conditional probabilities.--The length-of-cycle conditional probabilities are obtained by combining expressions (23) and (25) as follows:

$$P\{\Omega=\omega|x\} = \sum_{i=0}^{RP} \sum_{\nu=0}^{S-RP-1-i} G(i,\nu)P\{D(1) \geq S\text{-}RP\text{-}\nu\text{-}i\} \tag{26}$$
$$+ \sum_{i=0}^{RP} \sum_{h=1}^{i} \sum_{\nu=0}^{S-RP-1-i+h} G(i,h,\nu)P\{D(1) \geq S\text{-}RP\text{-}\nu\text{-}i\text{+}h\},$$

$$\text{where } G(i,\nu) = a(i)P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\} \tag{26a}$$

$$\text{and } G(i,h,\nu) = a(i)P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{26b}$$

$$\text{for } 1 \leq x+1 \leq K+1 \leq \omega.$$

Length-of-Cycle Unconditional Probabilities

The length-of-cycle unconditional probabilities can be obtained by inserting expression (26) into expression (21).

The expressions for the length-of-cycle probabilities developed in this section will be used (1) in the development of the stock level probabilities in the subsequent section and (2) in the determination of measures of effectiveness for the variable cycle inventory policy under the hypothesis of back-orders not allowed.

Stock Level Probabilities, Back-Orders Not Allowed

The objective of this section is to develop expressions for the stock level probabilities under the hypothesis of back-orders not allowed. These probabilities will be developed in terms of the demand and the replenishment lead time probability distributions; the order level, S; and the reorder point, RP. The beginning stock level probabilities are unconditional; however, the period stock level probabilities are developed upon the condition of a specific length of cycle.

Beginning Stock Level Stationary Probabilities

The probabilities that the beginning stock level is equal to i, a(i), after statistical equilibrium is attained, will be determined by solving a finite Markov transition matrix equation.

Markov transition matrix equation.--A transition from any beginning cycle stock level, i to any stock level, j, at the beginning of the subsequent variable length cycle, for $0 \leq i \leq RP$ and $0 \leq j \leq RP$, is possible as a consequence of the ordering rule. Therefore, the transition matrix, $M = (m_{ij})$, is regular; and the stationary probabilities exist and are unique. These stationary probabilities can be determined by solving the following matrix equation.

$$\underline{a} = \underline{a}\, M, \tag{27}$$

$$\text{where } \underline{a} = [a(RP),\ a(RP\text{-}1),\ .\ .\ .,\ a(0)], \tag{27a}$$

$$\text{and } M = \begin{bmatrix} m_{RP,RP} & m_{RP,RP-1} & \cdots & m_{RP,0} \\ m_{RP-1,RP} & m_{RP-1,RP-1} & \cdots & m_{RP-1,0} \\ \cdots & \cdots & \cdots & \cdots \\ m_{1,RP} & m_{1,RP-1} & \cdots & m_{1,0} \\ m_{0,RP} & m_{0,RP-1} & \cdots & m_{0,0} \end{bmatrix} . \tag{27b}$$

In the finite matrix, M, m_{ij} is the probability that the stock level is j items at the beginning of the subsequent variable length cycle, given that the stock level was i items at the beginning of the present variable length cycle.

Stock level transition probabilities.--The stock level transition probability, m_{ij}, will be obtained from the joint stock level transition probability elements, $m_{ij,\omega}$, for ω=K+1, K+2, ... The joint stock level transition element, $m_{ij,\omega}$, is the probability that the stock level at the beginning of the subsequent cycle is j items and that the number of periods in the variable length cycle is ω, given that the stock level was i items at the beginning of the present variable length cycle. The events $\Omega=\omega$, for ω=K+1, K+2, . . ., are mutually exclusive and exhaustive. Therefore, the joint stock level transition probabilities can be added. That is,

$$m_{ij} = \sum_{\omega=K+1}^{\infty} m_{ij,\omega} \qquad \text{for } 0 \le i \le RP \text{ and } 0 \le j \le RP. \tag{28}$$

The joint stock level transition probabilities, $m_{ij,\omega}$, will be developed from the joint stock level conditional transition probabilities, $m_{ij,\omega}|x$, corresponding to K+1 mutually exclusive and exhaustive sets, in conjunction with the probability of the specific replenishment lead time. That is,

$$m_{ij,\omega} = \sum_{x=0}^{K} (m_{ij,\omega}|x)P\{X=x\} \qquad \text{for } 0 \le i \le RP, \qquad (29)$$
$$0 \le j \le RP, \text{ and } K < \omega.$$

I. Joint stock level conditional transition probabilities.--The joint stock level conditional transition probability element, $m_{ij,\omega}|x$, is the probability that the stock level at the beginning of the subsequent cycle is j items and that the number of periods in the variable length cycle is ω, given that the stock level was i items at the beginning of the present cycle and that the replenishment lead time is x. The possible values for the replenishment lead time, X=x, and the beginning of the subsequent cycle stock level, j, will be divided into four mutually exclusive and exhaustive sets. The replenishment lead time is either zero or positive; and the beginning stock level of the subsequent cycle, j, is either zero or positive.

For a zero replenishment lead time, the stock level instantaneously attains the value of S.

For a positive replenishment lead time, there are two possible paths:

1. Demand during the replenishment lead time may be equal to or greater than the supply, $D(x) \ge i$, before the replenishment

order is received, in which case all demand in excess of supply is lost.

2. Demand during the replenishment lead time may be less than the supply, $D(x) < i$.

The four cases corresponding to the mutually exclusive and exhaustive sets are as follows:

Case I: Zero replenishment lead time ($x=0$) and zero stock at the beginning of the subsequent cycle ($j=0$).

Case II: Zero replenishment lead time ($x=0$) and a positive stock level at the beginning of the subsequent cycle ($j > 0$).

Case III: Positive replenishment lead time ($x > 0$) and a zero stock level at the beginning of the subsequent cycle.

Case IV: Positive replenishment lead time ($x > 0$) and a positive stock level at the beginning of the subsequent cycle ($j > 0$).

A. Case I: $x=0$ and $j=0$.--When the replenishment lead time is zero, S items of stock are instantaneously available. One path in which it is possible to have a transition to a zero supply at the beginning of the subsequent cycle occurs if demand during the ω-1 periods is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1$, and if demand during the next periods is equal to or greater than $S\text{-}\nu$. The probability of this specific path is

$$P\{D(\omega-1)=\nu\}P\{D(1) \geq S-\nu\} \quad \text{for } K < \omega \text{ and } 0 \leq \nu \leq S\text{-}RP\text{-}1. \qquad (30)$$

Figure 7 (page 42) illustrates a similar path for the fixed cycle inventory policy.

Demand during the ω-1 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore,

$$m_{ij,\omega}|0 = \sum_{\nu=0}^{S-RP-1} P\{D(\omega-1)=\nu\}P\{D(1) \geq S-\nu\} \tag{31}$$

for $0 \leq i \leq RP$, $j=0$, and $K < \omega$.

B. Case II: $x=0$ and $j > 0$.--Again, S items are instantaneously available since the replenishment lead time is zero. A transition to some positive supply of j items, for $1 \leq j \leq RP$, occurs if demand during the ω-1 periods is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1$, and if demand during the next period is equal to $S-\nu-j$. The probability of this specific path is

$$P\{D(\omega-1)=\nu\}P\{D(1)=S-\nu-j\} \quad \text{for } K < \omega,\ 0 \leq \nu \leq S\text{-}RP\text{-}1, \tag{32}$$

and $1 \leq j \leq RP$.

Figure 8 (page 42) illustrates a similar path for the fixed cycle inventory policy.

Demand during the ω-1 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore,

$$m_{ij,\omega}|0 = \sum_{\nu=0}^{S-RP-1} P\{D(\omega-1)=\nu\}P\{D(1)=S-\nu-j\} \tag{33}$$

for $K < \omega$, $0 \leq i \leq RP$, and $1 \leq j \leq RP$.

C. Case III: $x > 0$ and $j=0$.--Consider the two general paths corresponding to the mutually exclusive and exhaustive sets which terminate with $j=0$: $D(x) \geq i$ and $D(x) < i$.

1. Path I: $D(x) \geq i$.--When demand during the replenishment lead time is equal to or greater than i, the quantity of items available

prior to the receipt of the replenishment order is zero. Therefore, the quantity of items available when the replenishment order is received is S-i. A specific path in which there is a transition from a stock level of i items to a stock level of j items, for $0 \leq i \leq RP$ and $j=0$, occurs if the following three independent conditions exist:

1. Demand during the lead time is equal to or greater than i.
2. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1\text{-}i$.
3. Demand during period ω is equal to or greater than S-RP-ν-i.

For a specific value of ν, the probability of this path is

$$P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\}P\{D(1) \geq S-i-\nu\} \tag{34}$$
$$\text{for } 0 \leq i \leq RP,\ 1 \leq x \leq K < \omega, \text{and}$$
$$0 \leq \nu \leq S-RP-1-i.$$

Figure 9 (page 49) illustrates a similar path for the fixed cycle inventory policy.

Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1-i inclusive. Therefore, the probability of Path I is

$$\sum_{\nu=0}^{S-RP-1-i} G_1(\nu)P\{D(1) \geq S-i-\nu\}, \tag{35}$$

$$\text{where } G_1(\nu) = P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\}, \tag{35a}$$
$$\text{for } 1 \leq x \leq K < \omega.$$

2. Path II: $D(x) < i$.--When demand during the replenishment lead time is less than i, denote the quantity of items available prior to the receipt of the replenishment order as h. The quantity of items available after the replenishment order is received is h+S-i. A specific path in which there is a transition from a stock level of i items to a stock level of j items, for $0 \leq i \leq RP$ and j=0, occurs if the following three independent conditions exist:

1. Demand during the lead time is equal to i-h, which is less than i.
2. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S-RP-1-i+h$.
3. Demand during period ω is equal to or greater than $S-RP-i+h-\nu$.

For specific values of h and ν, the probability of this path is

$$P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}P\{D(1) \geq S-i+h-\nu\} \tag{36}$$

$$\text{for } 0 \leq h \leq i \leq RP,\ j=0,\ 1 \leq x \leq K < \omega, \text{ and}$$
$$0 \leq \nu \leq S-RP-1-i+h.$$

Figure 10 (page) illustrates a similar path for the fixed cycle inventory policy.

Demand during the replenishment lead time can be any of the possible values from 0 to i-1 inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1-i+h inclusive. Therefore, the probability of Path II is

$$\sum_{\nu=0}^{S-RP-1-i} \sum_{h=1}^{i} G(h,\nu)P\{D(1) \geq S-i+h-\nu\}, \tag{37}$$

$$\text{where } G(h,\nu) = P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}P\{D(1) \geq S-i+h-\nu\}, \tag{37a}$$

for $1 \leq x \leq K < \omega$, $0 \leq i \leq RP$, and $j=0$.

3. Summary of Case III.--The probability for Case III is obtained by combining expressions (35) and (37) as follows:

$$m_{ij,\omega}|x = \sum_{\nu=0}^{S-RP-1-i} G(\nu)P\{D(1) \geq S-i-\nu\} \tag{38}$$

$$+ \sum_{\nu=0}^{S-RP-1-i} \sum_{h=1}^{i} G(h,\nu)P\{D(1) \geq S-i+h-\nu\},$$

$$\text{where } G(\nu) = P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\} \tag{38a}$$

$$\text{and} \quad G(h,\nu) = P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{38b}$$

for $0 \leq i \leq RP$, $j=0$, and $1 \leq x \leq K < \omega$.

D. Case IV: $x > 0$ and $j > 0$.--Consider the two general paths corresponding to the mutually exclusive and exhaustive sets which terminate with $j > 0$: $D(x) \geq i$ and $D(x) < i$.

1. Path I: $D(x) \geq i$.--When demand during the replenishment lead time is equal to or greater than i, the quantity of items available prior to the receipt of the replenishment order is zero. The replenishment order is for S-i items. Therefore, the quantity of items available when the replenishment order is received is S-i. A specific path in

which there is a transition from a stock level of i items to a stock level of j items, for $0 \leq i \leq RP$ and $1 \leq j \leq RP$, occurs if the following three independent conditions exist:

1. Demand during the lead time is equal to or greater than i.
2. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1\text{-}i$.
3. Demand during period ω is equal to S-RP-ν-i-j.

For a specific value of ν, the probability of this path is

$$P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\}P\{D(1)=S-RP-\nu-i-j\} \tag{39}$$

$$\text{for } 0 \leq i \leq RP,\ 1 \leq j \leq RP,\ 1 \leq x \leq K < \omega,$$

$$\text{and } 0 \leq \nu \leq S-RP-1-i.$$

Figure 11 (page 49) illustrates a similar path for the fixed cycle inventory policy.

Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1-i inclusive. Therefore, the probability of Path I is

$$\sum_{\nu=0}^{S-RP-1-i} G(\nu)P\{D(1)=S-RP-i-\nu-j\}, \tag{40}$$

$$\text{where } G(\nu) = P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\}P\{D(1)=S-RP-i-\nu-j\}, \tag{40a}$$

$$\text{for } 0 \leq i \leq RP,\ 1 \leq j \leq RP, \text{ and } 1 \leq x \leq K < \omega.$$

2. Path II: $D(x) < i$.--When demand during the replenishment lead time is less than i, the quantity of items available prior to the receipt of the replenishment order will be denoted as h. The quantity of items available after the replenishment order is received is h+S-i.

A specific path in which there is a transition from a stock level of i items to a stock level of j items, for $0 \leq i \leq RP$ and $1 \leq j \leq RP$, occurs if the following three independent conditions exist:

1. Demand during the lead time is equal to i-h, which is less than i.
2. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1\text{-}i\text{+}h$.
3. Demand during period ω is equal to $S\text{-}RP\text{-}i\text{+}h\text{-}\nu\text{-}j$.

For specific values of h and ν, the probability of this path is

$$P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}P\{D(1)=S-RP-i+h-\nu-j\} \tag{41}$$

for $0 \leq h \leq i \leq RP,\ 1 \leq j \leq RP,\ 1 \leq x \leq K < \omega$, and $0 \leq \nu \leq S\text{-}RP\text{-}1\text{-}i\text{+}h$.

Figure 12 (page 49) illustrates a similar path for the fixed cycle inventory policy.

Demand during the replenishment lead time can be any of the possible values from 0 to i-1 inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1-i+h inclusive. Therefore, the probability of Path II is

$$\sum_{\nu=0}^{S-RP-1-i}\ \sum_{h=1}^{i} G(h,\nu)P\{D(1)=S-RP-i+h-\nu-j\}\ , \tag{42}$$

$$\text{where } G(h,\nu) = P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{42a}$$

for $1 \leq x \leq K < \omega$ and $1 \leq j \leq RP$.

3. Summary of Case IV.--The probability for Case IV is obtained by combining expressions (40) and (42) as follows:

$$m_{ij,\omega}|x = \sum_{\nu=0}^{S-RP-1-i} G(\nu)P\{D(1)=S-RP-i+h-\nu-j\} \tag{43}$$

$$+ \sum_{\nu=0}^{S-RP-1-i} \sum_{h=1}^{i} G(h,\nu)P\{D(1)=S-RP-i+h-\nu-j\},$$

$$\text{where } G(\nu) = P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\} \tag{43a}$$

$$\text{and} \quad G(h,\nu) = P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{43b}$$

for $0 \leq i \leq RP$, $1 \leq j \leq RP$, and $1 \leq x \leq K < \omega$.

E. Summary of the joint stock level conditional transition probabilities.--Expressions (31), (33), (38), and (43) are summarized as follows: (see next page)

$$
m_{ij,\omega}|x = \begin{cases}
\displaystyle\sum_{\nu=0}^{S-RP-1} P\{D(\omega-1)=\nu\}P\{D(1) \geq S-\nu\} & \\
\qquad \text{for } 0 \leq i \leq RP,\ j=0,\ x=0, \text{ and } K < \omega; & \\[2ex]
\displaystyle\sum_{\nu=0}^{S-RP-1} P\{D(\omega-1)=\nu\}P\{D(1)=S-\nu-j\} & \\
\qquad \text{for } 0 \leq i \leq RP,\ 1 \leq j \leq RP,\ x=0, \text{ and } K < \omega; & \\[2ex]
\displaystyle\sum_{\nu=0}^{S-RP-1-i} G(\nu)P\{D(1) \geq S-i+h-\nu\} & \\
\displaystyle\quad + \sum_{\nu=0}^{S-RP-1-i} \sum_{h=1}^{i} G(h,\nu)P\{D(1) \geq S-i+h-\nu\} & \\
\qquad \text{for } 0 \leq i \leq RP,\ j=0, \text{ and } 1 \leq x \leq K < \omega; & \\[2ex]
\displaystyle\sum_{\nu=0}^{S-RP-1-i} G(\nu)P\{D(1)=S-RP-i+h-\nu-j\} & \\
\displaystyle\quad + \sum_{\nu=0}^{S-RP-1-i} \sum_{h=1}^{i} G(h,\nu)P\{D(1)=S-RP-i+h-\nu-j\} & \\
\qquad \text{for } 0 \leq i \leq RP,\ 1 \leq j \leq RP, \text{ and } 1 \leq x \leq K < \omega; &
\end{cases} \tag{44}
$$

where $G(\nu) = P\{D(x) \geq i\}P\{D(\omega-x-1)=\nu\}$ (44a)

and $G(h,\nu) = P\{D(x)=i-h\}P\{D(\omega-x-1)=\nu\}.$ (44b)

II. Stock level unconditional transition probabilities.--The stock level unconditional transition probabilities can be obtained by inserting expression (44) into expression (29) and then inserting this expression into expression (28).

The stationary probabilities for the stock level at the beginning of the variable length order cycle can be determined from expression (27) utilizing the transition probabilities obtained from expressions (44), (29), and (28).

Period Stock Level Conditional Stationary Probabilities

The conditional stationary probabilities that the stock level is equal to h prior to the demand in period r, given ω periods in the cycle, are denoted by $a_r(h|\omega)$. These probabilities will be developed from the period stock level double conditional probabilities, $a_r(h|\omega,x)$, corresponding to K+1 mutually exclusive and exhaustive sets, in conjunction with the conditional probability of the lead time for a given length of cycle. That is,

$$a_r(h|\omega) = \sum_{x=0}^{K} [a_r(h|\omega,x)]P\{X=x|\omega\} \qquad (45)$$

for $1 \leq r \leq N$, $K < \omega$, and $0 \leq h \leq S$.

Under the hypothesis of back-orders not allowed, X and Ω are not independent. The following identity will be useful in the subsequent development:

$$P\{X=x|\omega\} = \frac{P\{\Omega=\omega|x\}P\{X=x\}}{P\{\Omega=\omega\}} . \qquad (46)$$

By inserting expression (46) into expression (45), a convenient expression is obtained:

$$a_r(h|\omega) = \frac{1}{P\{\Omega=\omega\}} \sum_{x=0}^{K} [a_r(h|\omega,x)]P\{\Omega=\omega|x\}P\{X=x\} \qquad (47)$$

for $1 \leq r \leq N$, $K < \omega$ and $0 \leq h \leq S$.

Period stock level double conditional stationary probabilities.--The double conditional probabilities that the stock level is equal to h prior to the demand in period r, given that ω periods are in the order cycle and given that the replenishment lead time is equal to x periods for the beginning cycle order, are denoted by $a_r(h|\omega,x)$. These probabilities will be developed from the following expression:

$$a_r(h|\omega,x) = \frac{P\{H_r=h,\ \Omega=\omega|X=x\}}{P\{\Omega=\omega|X=x\}}, \qquad (48)$$

where H_r is the stock level random variable at period r and h=0, 1, . . ., S are the values which this random variable assumes. The denominator of expression (48) has been previously developed as expression (26). In this development of the numberator of expression (39a), consider the mutually exclusive and exhaustive sets: $r < x+1$ and $r \geq x+1$.

I. Case I: $r < x+1$.--The joint probabilities that the stock level is equal to h prior to the demand in period r and that the length of cycle is ω periods will be developed. Consider the sub-cases corresponding to the sub-sets obtained by trichotomizing this set into the following mutually exclusive and exhaustive sub-sets: h=0, $1 \leq h \leq RP$, and $RP < h \leq S$.

A. <u>Sub-case I: h=0</u>.--A specific path in which the stock level is equal to zero prior to the demand in period r, $1 \leq r < x+1 \leq K+1$ and in which the length of cycle is equal to ω occurs if the following four independent conditions exist:

1. The beginning stock level is equal to i.
2. Demand during the first r-1 periods is equal to or greater than i.
3. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}i\text{-}1$.
4. Demand during period ω is equal to or greater than S-RP-i-ν.

For specific values of i and ν, the probability of this path is

$$a(i)P\{D(r-1) \geq i\}P\{D(\omega-x-1)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}i\text{-}\nu\} \tag{49}$$

for $0 \leq i \leq RP$, $1 \leq r < x+1 \leq K+1 \leq \omega$, and

$0 \leq \nu \leq S\text{-}RP\text{-}i\text{-}1$.

The beginning stock level can be any of the possible values from 0 to RP inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-i-1 inclusive. Therefore,

$$P\{H_r=0, \Omega=\omega | X=x\} \tag{50}$$

$$= \sum_{i=0}^{RP} \sum_{\nu=0}^{S\text{-}RP\text{-}i\text{-}1} G_1(i,\nu)P\{D(r-1) \geq i\}P\{D(1) \geq S\text{-}RP\text{-}i\text{-}\nu\},$$

$$\text{where } G_1(i,\nu) = a(i)P\{D(\omega-x-1)=\nu\}, \tag{50a}$$

for $1 \leq r < x+1 \leq K+1 \leq \omega$.

B. <u>Sub-case II: $1 \leq h \leq RP$</u>.--Consider the two general paths corresponding to the mutually exclusive and exhaustive sets in which $D(x) \geq i$ and $D(x) < i$.

1. <u>Path I: $D(x) \geq i$</u>.--A specific path in which the stock level is equal to h, for $1 \leq h \leq RP$, prior to the demand in period r, for $1 \leq r < x+1 \leq K+1$, and in which the length of cycle is equal to ω occurs if the following five independent conditions exist:

1. The beginning stock level is equal to i.
2. Demand during the first r-1 periods is equal to i-h.
3. Demand during the next x+1-r periods is equal to or greater than h.
4. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}i\text{-}1$.
5. Demand during period ω is equal to or greater than $S\text{-}RP\text{-}i\text{-}\nu$.

For specific values of i and ν, the probability of this path is

$$G_2(i,\nu)P\{D(x+1-r) \geq h\}P\{D(1) \geq S\text{-}RP\text{-}i\text{-}\nu\}, \tag{51}$$

$$\text{where } G_2(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{51a}$$

$$\text{for } 1 \leq h \leq i \leq RP,\ 1 \leq r < x+1 \leq K+1 \leq \omega, \text{ and}$$
$$0 \leq \nu \leq S\text{-}RP\text{-}i\text{-}1.$$

The beginning stock level can be any of the possible values from h to S inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-i-1 inclusive. Therefore,

$$P\{H_r=h, \Omega=\omega, D(x) \geq i \mid X=x\} \tag{52}$$

$$= \sum_{i=h}^{RP} \sum_{\nu=0}^{S-RP-i-1} G_2(i,\nu)P\{D(x+1-r) \geq h\}P\{D(1) \geq S-RP-i-\nu\},$$

$$\text{where } G_2(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{52a}$$

$$\text{for } 1 \leq r < x+1 \leq K+1 \leq \omega \text{ and } 1 \leq h \leq RP.$$

2. Path II: $D(x) < i$.--A specific path in which the stock level is equal to h, for $1 \leq h \leq RP$, prior to the demand in period r, for $1 \leq r < x+1 \leq K+1$, and in which the length of cycle is equal to ω occurs if the following independent five conditions exist:

1. The beginning stock level is equal to i.
2. Demand during the first r-1 periods is equal to i-h.
3. Demand during the next x+1-r periods is equal to ξ, for $0 \leq \xi \leq h-1$.
4. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq S-RP-i+h-\xi-1$.
5. Demand during period ω is equal to or greater than $S-RP-i+h-\xi-\nu$.

For specific values of i, ξ, and ν, the probability of this path is

$$G_2(i,\nu)P\{D(x+1-r)=\xi\}P\{D(1) \geq S-RP-i+h-\xi-\nu\}, \tag{53}$$

$$\text{where } G_2(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-x-1)=\nu\}, \tag{53a}$$

$$\text{for } 0 \leq i \leq RP,\ 1 \leq r < x+1 \leq K+1 \leq \omega,\ 0 \leq \nu \leq S-RP-i+h-\xi-1,$$

$$0 \leq \xi \leq h-1, \text{ and } 1 \leq h \leq RP.$$

The beginning stock level can be any of the possible values from h to S inclusive. Demand during the x+1-r periods can be any of the possible values from 0 to h-1 inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-i+h-ξ-ν inclusive. Therefore,

$$P\{H_r=h, \Omega=\omega, D(x) < i \mid X=x\} \tag{54}$$

$$= \sum_{i=h}^{RP} \sum_{\xi=0}^{h-1} \sum_{\nu=0}^{S-RP-i+h-\xi-1} G_2(i,\nu)P\{D(x+1-r)=\xi\}P\{D(1) \geq S-RP-i+h-\xi-\nu\},$$

where $G_2(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-x-1)=\nu\}$, (54a)

for $1 \leq r < x+1 \leq K+1 \leq \omega$ and $1 \leq h \leq RP$.

3. Summary of Sub-case II.--The probability for Sub-case II is obtained by combining expressions (52) and (54) as follows:

$$P\{H_r=h, \Omega=\omega \mid X=x\} \tag{55}$$

$$= \sum_{i=h}^{RP} \sum_{\nu=0}^{S-RP-i-1} G_2(i,\nu)P\{D(x+1-r) \geq h\}P\{D(1) \geq S-RP-i-\nu\}$$

$$+ \sum_{i=h}^{RP} \sum_{\xi=0}^{h-1} \sum_{\nu=0}^{S-RP-i+h-\xi-1} G_2(i,\nu)P\{D(x+1-r)=\xi\}P\{D(1) \geq S-RP-i+h-\xi-\nu\},$$

where $G_2(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-x-1)=\nu\}$, (55a)

for $1 \leq r < x+1 \leq K+1 \leq \omega$ and $1 \leq h \leq RP$.

C. Sub-case III: $RP < h \leq S$.--It is impossible that the stock level at the beginning of period r, for $1 \leq r < x+1 \leq K+1$, is greater than RP. Therefore,

$$P\{H_r=h,\Omega=\omega|X=x\} = 0 \tag{56}$$

for $1 \leq r < x+1 \leq K+1 \leq \omega$ and $RP < h \leq S$.

II. Case II: $r \geq x+1$.--The joint probabilities that the stock level is equal to h prior to the demand in period r and that the length of cycle is ω periods, given that the lead time is equal to x, will be developed. Consider the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $0 \leq h \leq RP$ and $RP < h \leq S$.

A. Sub-case I: $0 \leq h \leq RP$.--If the stock level is equal to or less than RP, a replenishment order has been placed. Since the placement of a replenishment order constitutes the beginning of an order cycle, the period is considered as being in the subsequent variable order cycle. Therefore,

$$P\{H_r=h,\Omega=\omega|X=x\} = 0 \tag{57}$$

for $1 \leq x+1 \leq r \leq \omega$ and $0 \leq h \leq RP$.

B. Sub-case II: $RP < h \leq S$.--Consider the two general paths corresponding to the mutually exclusive and exhaustive sets in which $D(x) \geq i$ and $D(x) < i$.

A. Path I: $D(x) \geq i$.--A specific path in which the stock level is equal to h, for $RP \leq h \leq S$, prior to the demand in period r, for $1 < x+1 \leq r \leq \omega$, and in which the length of cycle is equal to ω occurs by Path I if the following five independent conditions exist:

1. The beginning stock level is equal to $S-h-\xi$.
2. Demand during the replenishment lead time is equal to or greater than $S-h-\xi$.

3. Demand during the r-x periods following receipt of the replenishment order is equal to ξ, for $0 \leq \xi \leq S\text{-}h$.

4. Demand during the next ω-r periods is equal to ν, for $0 \leq \nu \leq h\text{-}RP\text{-}1$.

5. Demand during period ω is equal to or greater than h-RP-ν.

For a specific value of ν, the probability of this path is

$$G_3(\xi)P\{D(\omega-r)=\nu\}P\{D(1) \geq h-RP-\nu\}, \tag{58}$$

$$\text{where } G_3(\xi) = a(S-h-\xi)P\{D(x) \geq S-h-\xi\}P\{D(r-x)=\xi\}, \tag{58a}$$

$$\text{for } 1 \leq x+1 \leq r \leq \omega,\ 0 \leq \nu \leq h-RP-1, \text{ and } RP \leq h \leq S.$$

Demand during the r-x periods can be any of the possible values from 0 to S-h. Demand during the ω-r periods can be any of the possible values from 0 to h-RP-1 inclusive. Therefore,

$$P\{H_r=h, \Omega=\omega, D(x) > i \mid X=x\} \tag{59}$$

$$= \sum_{\xi=0}^{S-h} \sum_{\nu=0}^{h-RP-1} G_3(\xi)P\{D(\omega-r)=\nu\}P\{D(1) \geq h-RP-\nu\},$$

$$\text{where } G_3(\xi) = a(S-h-\xi)P\{D(x) \geq S-h-\xi\}P\{D(r-x)=\xi\}, \tag{59a}$$

$$\text{for } 1 \leq x+1 \leq r \leq \omega \text{ and } RP \leq h \leq S.$$

B. <u>Path II: D(x) < i</u>.--A specific path in which the stock level is equal to h, for $RP \leq h \leq S$, prior to the demand in period r, for $1 \leq x+1 \leq r \leq \omega$, and in which the length of cycle is equal to ω occurs by Path II if the following four independent conditions exist:

1. The beginning stock level is equal to i, for $S\text{-}h\text{+}1 \leq i \leq S$.

2. Demand during the replenishment lead time is equal to S-h.

3. Demand during the ω-x-1 periods following receipt of the replenishment order is equal to ν, for $0 \leq \nu \leq$ h-RP-1.

4. Demand during period ω is equal to or greater than h-RP-ν.

For specific values of i and ν, the probability of this path is

$$a(i)P\{D(x)=S-h\}P\{D(\omega-x-1)=\nu\}P\{D(1) \geq h-RP-\nu\} \tag{60}$$

$$\text{for } 1 \leq x+1 \leq r \leq \omega,\ 0 \leq \nu \leq h-RP-1,$$

$$\text{and } RP \leq h \leq S.$$

The beginning stock level can be any of the possible values from S-h+1 to S inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to h-RP-1 inclusive. Therefore,

$$P\{H_r=h, \Omega=\omega, D(x) < i \mid X=x\} \tag{61}$$

$$= \sum_{i=S-h+1}^{S} \sum_{\nu=0}^{h-RP-1} G_4 a(i)P\{D(\omega-x-1)=\nu\}P\{D(1) \geq h-RP-\nu\},$$

$$\text{where } G_4 = P\{D(x)=S-h\}, \tag{61a}$$

$$\text{for } 1 \leq x+1 \leq r \leq \omega \quad \text{and} \quad RP \leq h \leq S.$$

C. <u>Summary of Case II</u>.--The probability for Sub-case II is obtained by combining expressions (59) and (61) as follows:

$$P\{H_r=h,\Omega=\omega|X=x\} \tag{62}$$

$$= \sum_{\xi=0}^{S-h} \sum_{\nu=0}^{h-RP-1} G_3(\xi)P\{D(\omega-r)=\nu\}P\{D(1) \geq h-RP-\nu\}$$

$$+ \sum_{i=S-h+1}^{S} \sum_{\nu=0}^{h-RP-1} G_4 a(i)P\{D(\omega-x-1)=\nu\}P\{D(1) \geq h-RP-\nu\},$$

where $G_3(\xi) = a(S-h-\xi)P\{D(x) \geq S-h-\xi\}P\{D(r-x)=\xi\}$ (62a)

and $G_4 = P\{D(x)=S-h\}$,

for $1 < x+1 \leq r \leq K < \omega$ and $RP \leq h \leq S$.

III. Summary of period stock level double conditional stationary probabilities.--Expressions (50), (55), (56), (57), and (62) are inserted into expression (40) and are summarized as follows: (see next page)

$$a_r(h|\omega,x) = \begin{cases} \dfrac{\sum\limits_{i=0}^{RP} \sum\limits_{\nu=0}^{S-RP-i-1} G_1(i,\nu)P\{D(r-1) \geq i\}P\{D(1) \geq S-RP-i-\nu\}}{P\{\Omega=\omega|x\}} & \text{for } 1 \leq r < x+1 \leq K+1 \leq \omega \text{ and } h=0, \\[2ex] \dfrac{\sum\limits_{i=h}^{RP} \sum\limits_{\nu=0}^{S-RP-i-1} G_2(i,\nu)P\{D(x+1-r) \geq h\}P\{D(1) \geq S-RP-i-\nu\}}{P\{\Omega=\omega|x\}} \\ \quad + \dfrac{\sum\limits_{i=h}^{RP} \sum\limits_{\xi=0}^{h-1} G_2(i,\nu)P\{D(x+1-r)=\xi\}P\{D(1) \geq S-RP-i+h-\xi-\nu\}}{P\{\Omega=\omega|x\}} & \text{for } 1 \leq r < x+1 \leq K+1 \leq \omega \text{ and } 1 \leq h \leq RP, \\[2ex] 0 & \text{for } 1 \leq r < x+1 \leq K+1 \leq \omega \text{ and } RP < h \leq S, \\[2ex] 0 & \text{for } 1 \leq x+1 \leq r \leq \omega \text{ and } 0 \leq h \leq RP, \\[2ex] \dfrac{\sum\limits_{\xi=0}^{S-h} \sum\limits_{\nu=0}^{h-RP-1} G_3(\xi)P\{D(\omega-r)=\nu\}P\{D(1) \geq h-RP-\nu\}}{P\{\Omega=\omega|x\}} \\ \quad + \dfrac{\sum\limits_{i=S-h+1}^{S} \sum\limits_{\nu=0}^{h-RP-1} G_4a(i)P\{D(\omega-x-1)=\nu\}P\{D(1) \geq h-RP-\nu\}}{P\{\Omega=\omega|x\}} & \text{for } 1 \leq x+1 \leq r \leq \omega \text{ and } RP < h \leq S, \end{cases} \quad (63)$$

where $G_1(i,\nu) = a(i)P\{D(\omega-x-1)=\nu\}$, (63a)

$$G_2(i,\nu) = a(i)P\{D(r-1)=i-h\}P\{D(\omega-x-1)=\nu\}, \quad (63b)$$

$$G_3(\xi) = a(S-h-\xi)P\{D(x) \geq S-h-\xi\}P\{D(r-x)=\xi\}, \qquad (63c)$$

and $$G_4 = P\{D(x)=S-h\}. \qquad (63d)$$

Period stock level conditional stationary probabilities.--The period stock level conditional probabilities, given that the length of cycle is ω periods, $a_r(h|\omega)$, can be obtained by inserting expressions (63), (26), and (21) into expression (47).

The beginning stock level stationary probabilities and the period stock level conditional probabilities developed in this section will be used as the basis for determining the measures of effectiveness for the variable cycle inventory policy under the hypothesis that back-orders are not allowed.

Results

The expressions for the length-of-cycle probabilities and the stock level probabilities for the variable cycle inventory policy have been developed under the hypothesis of back-orders allowed and the hypothesis of back-orders not allowed. The results of this chapter will be used in Chapter VII in the determination of the measures of effectiveness required to attain the primary objective of the study.

CHAPTER VI

STOCK LEVEL PROBABILITIES FOR THE COMBINATION INVENTORY POLICY

Introduction

The objective of this chapter is to develop mathematical expressions for the stock level probabilities in terms of relevant controlled decision variables characteristic of the combination inventory policy and uncontrolled random variables. This combination inventory policy requires that a replenishment order be placed at the beginning of a cycle composed of a fixed number of periods and also at the beginning of the first period within the first Λ periods of the cycle in which the sum of stock level and stock on order is equal to or less than the reorder point. Both the beginning-cycle and the within-the-cycle replenishment orders are for a quantity equal to the difference between the order level and the sum of stock level and stock on order. The relevant controlled decision variables are the order level, S; the reorder point, RP; the number of periods, Λ, within which the within-the-cycle replenishment order may be placed; and the number of periods, N, in the order cycle. The uncontrolled random variables are the demand and the replenishment lead time. The stock level probabilities will be developed only for the hypothesis of back-orders allowed.

The literature search, reported in Chapter II, discloses no reference in which analytical developments are cited pertaining to this or any other similar combination inventory policy. This treatment of the combination inventory policy will consider the demand and the replenishment lead time as random variables obtained from any probability distribution. The assumptions listed and discussed in Chapter I will be used as a basis for the analytical development of the combination inventory policy considered in this chapter. In addition, the replenishment lead time is assumed to be equal to or less than N-1 periods.

Stock Level Probabilities

The objective of this section is to develop expressions for the stock level probabilities. These expressions will be developed under the hypothesis of back-orders allowed in terms of the demand and the replenishment lead time probability distributions; the order level, S; the reorder point, RP; the number of periods, Λ, within which the within-the-cycle replenishment order may be placed; and the number of periods, N, in the order cycle.

When back-orders are allowed, unsatisfied demands are deferred either until the next replenishment order is received or until sufficient stock is available. Under the combination inventory policy a replenishment order is placed at the beginning of each order cycle and also at the end of the first period within the cycle in which the sum of stock level and stock on order is equal to or below the reorder point, RP. The quantity of items ordered for both of these types of replenishment orders depends either upon (1) the difference between the order level, S, and the sum of stock level and stock-on-order or upon (2) the sum of order level, S, and unsatisfied demand minus stock-on-order. That is,

$$\text{replenishment order} = \begin{cases} S - (\text{stock available} + \text{stock-on-order}) & (1) \\ \qquad \text{for a positive stock level} \\ S + (\text{unsatisfied demand} - \text{stock-on-order}) \\ \qquad \text{for a non-positive stock level.} \end{cases}$$

In the development of the stock level probabilities, it is convenient to define the sets illustrated in Figure 17 (page 127) as follows:

1. Set A: Contains all of the paths in which a within-the-cycle replenishment order is placed within the first Λ periods of the order cycle.

2. Set $\bar{\Lambda}$: Contains all of the paths in which a within-the-cycle replenishment order is not placed within the first Λ periods of the order cycle.

3. Set B: Contains all of the paths in which the within-the-cycle replenishment order is received within the same order cycle in which the replenishment order is placed.

4. Set $\bar{B}$: Contains all of the paths in which a within-the-cycle replenishment order is not received within the same order cycle. Either a within-the-cycle replenishment order is not placed, or a within-the-cycle replenishment order is placed and not received until during the subsequent order cycle.

5. Set E: Contains all of the paths in which the cumulative demand during the entire order cycle is less than S-RP.

6. Set $\bar{E}$: Contains all of the paths in which the cumulative demand during the entire order cycle is equal to or greater than S-RP.

7. Set $\bar{r}$: Contains all of the paths in which the carry-over replenishment order is received at the beginning of period $\bar{r}$.

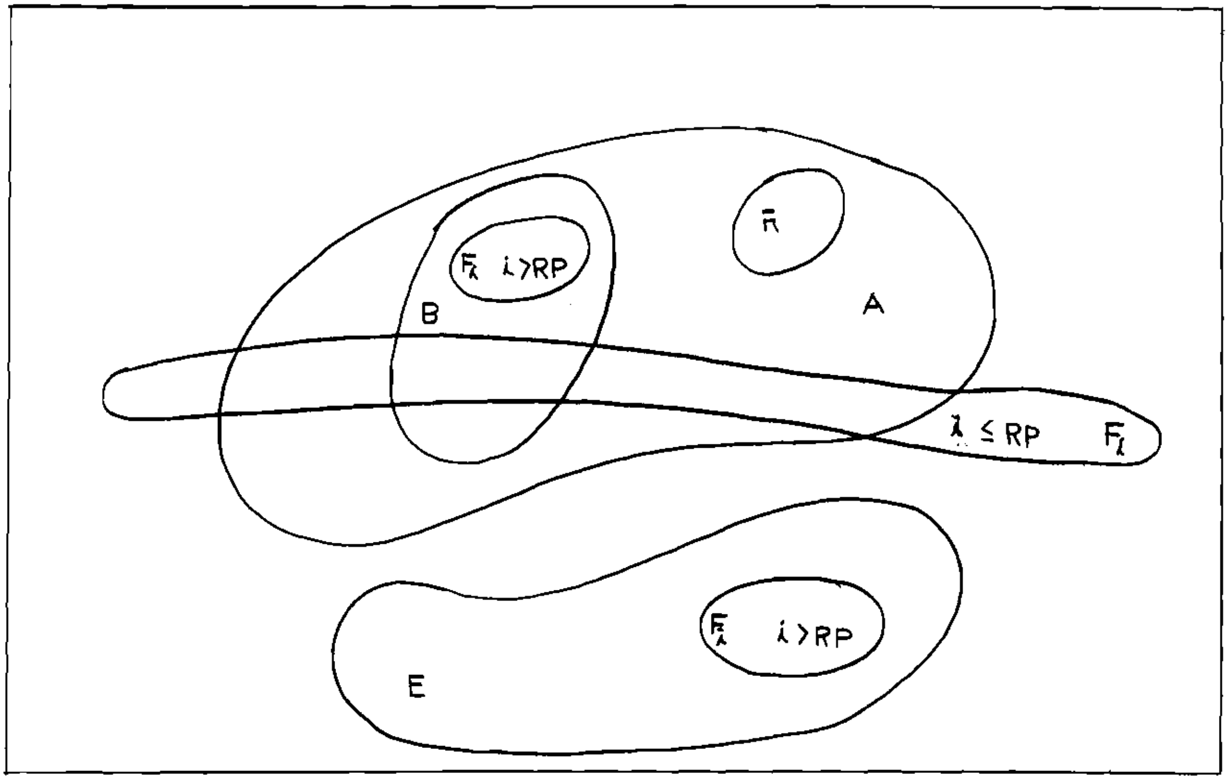

Fig. 17. Venn Diagram Illustrating the Relationships Between the Sets A, B, and E, Corresponding to the Combination Inventory Policy

8. Set F_i: Contains the paths in which the beginning stock level is i items.

Only those sets that are possible are shown in the Venn diagram in Figure 17 (page 127).

Beginning Stock Level Conditional Stationary Probabilities

In the subsequent development it is convenient to utilize four beginning stock level conditional stationary probabilities: $a(i|\bar{A})$, $a(i|AB)$, $a(i|A\bar{B})$ and $a(i|A\bar{B},r)$. These probabilities represent the probability that the beginning stock level is equal to i, given the sets $\bar{A}$, AB, $A\bar{B}$ and $A\bar{B},\bar{r}$, respectively. The sets $\bar{A}$, AB and $A\bar{B}$ are as previously defined; and the set $A\bar{B},\bar{r}$ is the intersection of the set $A\bar{B}$ with $\bar{r}$.

Stock level conditional probabilities, $a(i|\bar{A})$.--The conditional probabilities that the beginning stock level is equal to i, given that a within-the-cycle replenishment order was not placed during the previous cycle, are denoted by $a(i|\bar{A})$. These probabilities will be determined by considering the cases corresponding to two mutually exclusive and exhaustive sets: $i \leq RP$ and $RP < i \leq S$.

I. Case I: $i \leq RP$.--The beginning stock level, i, for $i \leq RP$ can be obtained only by the paths contained in the set

$$\bar{A}\,\bar{E}\,F_i. \tag{2}$$

Therefore,

$$a(i|\bar{A}) = \frac{P\{\bar{A}\,\bar{E}\,F_i\}}{P\{\bar{A}\}} \text{ for } i \leq RP. \tag{3}$$

The joint probability that a replenishment order is not placed during the first Λ periods and that the stock level at the beginning of the subsequent cycle is equal to i, for $i \leq RP$, will be determined. A specific path in the set $\overline{A}\,\overline{E}\,F_i$ occurs if the cumulative demand during the first Λ-1 periods is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1$, and if demand during the remaining periods is equal to $S\text{-}i\text{-}\nu$. The probability of this specific path is

$$P\{D(\Lambda-1)=\nu\}P\{D(N-\Lambda+1)=S-i-\nu\} \qquad \text{for } 0 \leq \nu \leq S\text{-}RP\text{-}1, \tag{4}$$
$$2 \leq \Lambda \leq N, \text{ and } i \leq RP.$$

Since demand during the Λ-1 periods can be any of the possible values from 0 to S-RP-1 inclusive,

$$P\{\overline{A}\,\overline{E}\,F_i\} = \sum_{\nu=0}^{S-RP-1} P\{D(\Lambda-1)=\nu\}P\{D(N-\Lambda+1)=S-i-\nu\} \tag{5}$$
$$\text{for } i \leq RP \text{ and } 2 \leq \Lambda \leq N.$$

II. <u>Case II: $RP < i \leq S$.</u>--The beginning stock level of the subsequent cycle, i for $RP < i \leq S$, can be obtained only by the paths contained in the set

$$\overline{A}\,EF_i. \tag{6}$$

Therefore,

$$a(i|\overline{A}) = \frac{P\{\overline{A}\,EF_i\}}{P\{\overline{A}\}} \qquad \text{for } RP < i \leq S.$$

The joint probability that a replenishment order is not placed during the first Λ periods and that the stock level at the beginning of the subsequent cycle is equal to i, for RP < i ≤ S, will be determined. A specific path in the set $\overline{A}EF_i$ occurs if the cumulative demand during the first Λ-1 periods is equal to ν, for $0 \leq \nu \leq S-i$, and if demand during the remaining periods is equal to $S-i-\nu$. The probability of this specific path is

$$P\{D(\Lambda-1)=\nu\}P\{D(N-\Lambda+1)=S-i-\nu\} \tag{8}$$

$$\text{for } 0 \leq \nu \leq S-i,\ 2 \leq \Lambda \leq N, \text{ and } RP < i \leq S.$$

Since demand during the first Λ-1 periods can be any of the possible values from 0 to S-i inclusive,

$$P\{\overline{A}EF_i\} = \sum_{\nu=0}^{S-i} P\{D(\Lambda-1)=\nu\}P\{D(N-\Lambda+1)=S-i-\nu\} \tag{9}$$

$$\text{for } RP < i \leq S \text{ and } 2 \leq \Lambda \leq N.$$

Since $D(\Lambda-1)$ and $D(N-\Lambda+1)$ are non-negative independent integral valued random variables (11, p. 250), expression (9) is a convolution and can be expressed as

$$P\{\overline{A}EF_i\} = P\{D(N)=S-i\} \qquad \text{for } RP < i \leq S. \tag{10}$$

III. <u>Probability of the condition, $\overline{A}$.</u>--The probability that a within-the-cycle replenishment order is not placed will be determined. A specific path in the set $\overline{A}$ occurs if demand during the Λ-1 periods is equal to ν, for $0 \leq \nu \leq S-RP-1$. The probability of this path is

$$P\{D(\Lambda-1)=\nu\} \qquad \text{for } 0 \leq \nu \leq S-RP-1 \quad \text{and} \quad 2 \leq \Lambda \leq N. \tag{11}$$

Since demand during the first Λ-1 periods can be any of the possible values from 0 to S-RP-1 inclusive,

$$P\{\bar{A}\} = \sum_{\nu=0}^{S\text{-}RP\text{-}1} P\{D(\Lambda-1)=\nu\} = P\{D(\Lambda-1) \leq S\text{-}RP\text{-}1\} \qquad (12)$$

$$\text{for } 2 \leq \Lambda \leq N.$$

IV. Summary of the stock level conditional probabilities, $a(i|\bar{A})$.--The stock level conditional probabilities, $a(i|\bar{A})$, are obtained by dividing expression (5) by expression (12) and by dividing expression (10) by expression (12). These probabilities are summarized, for $i \leq S$ and $2 \leq \Lambda \leq N$, as follows:

$$a(i|\bar{A}) = \begin{cases} \dfrac{1}{G} \displaystyle\sum_{\nu=0}^{S\text{-}RP\text{-}1} P\{D(\Lambda-1)=\nu\}P\{D(N-\Lambda+1)=S-i-\nu\} & \text{for } i \leq RP, \\ \dfrac{1}{G}P\{D(N)=S-i\} & \text{for } RP < i \leq S, \end{cases} \qquad (13)$$

$$\text{where } G = P\{D(\Lambda-1) \leq S\text{-}RP\text{-}1\}. \qquad (13a)$$

Stock level conditional probabilities, $a(i|AB)$.--The conditional probabilities, that the beginning stock level is equal to i, given that a within-the-cycle replenishment order was placed and received during the previous cycle are denoted by $a(i|AB)$. These probabilities will be determined from the expression

$$a(i|AB) = \frac{P\{ABF_i\}}{P\{AB\}} \qquad \text{for } i \leq S. \qquad (14)$$

I. Probability of the set ABF_i.--The joint probability that a replenishment order is placed at the beginning of any of the periods during the first Λ periods, that this within-the-cycle replenishment order is received during the same order cycle, and that the stock level at the beginning of the subsequent cycle is equal to i will be determined. A specific path in the set ABF_i occurs if the following independent conditions exist:

1. The cumulative demand at the beginning of any period during the first Λ periods of the order cycle is equal to or exceeds S-RP, causing a within-the-cycle replenishment order to be placed. This within-the-cycle replenishment order is placed at the beginning of period r if the cumulative demand during the first r-2 periods is equal to ν, which is less than S-RP, and if demand during the first r-1 periods is equal to or greater than S-RP.
2. This within-the-cycle replenishment order is received within N-r periods.
3. Demand during the N-r+1 periods after the replenishment order is placed is equal to S-i.

For specific values of r and ν, the probability of this path is

$$P\{D(r-2)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}\nu\}P\{D(N\text{-}r\text{+}1)=S\text{-}i\}P\{X \leq N\text{-}r\} \qquad (15)$$

$$\text{for } 2 \leq r \leq \Lambda \leq N,\ 0 \leq \nu \leq S\text{-}RP\text{-}1, \text{ and } i \leq S.$$

If the within-the-cycle replenishment order is placed at the beginning of period r, the demand during the r-2 periods can be any of the possible values from 0 to S-RP-1 inclusive. Since the within-the-cycle replenishment order can be placed during any of the first Λ periods except the first, r can be any of the possible values from 2 to Λ inclusive. Therefore,

$$P\{ABF_i\} = \sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(r-2)=\nu\}P\{D(N-r+1)=S-i\}G_1(r,\nu), \qquad (16)$$

$$\text{where } G_1(r,\nu) = P\{D(1) \geq S\text{-}RP\text{-}\nu\}P\{X \leq N\text{-}r\}, \qquad (16a)$$

$$\text{for } 2 \leq \Lambda \leq N \text{ and } i \leq S.$$

II. Probability of the condition, AB.--The joint probability that a within-the-cycle replenishment order is placed and received during the same order cycle will be determined. A specific path in the set AB occurs if the first two conditions specified for the set ABF_i exist. The probability of this specific path is

$$P\{D(r-2)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}\nu\}P\{X \leq N\text{-}r\} \qquad (17)$$

$$\text{for } 2 \leq r \leq \Lambda \leq N \text{ and } 0 \leq \nu \leq S\text{-}RP\text{-}1.$$

If the within-the-cycle replenishment order is placed at the end of period r, demand during the first r-2 periods can be any of the possible values from 0 to S-RP-1 inclusive. Since the within-the-cycle replenishment order can be placed during any of the first Λ periods except the first, r can be any of the possible values from 2 to Λ inclusive. Therefore,

$$P\{AB\} = \sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(r-2)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}\nu\}P\{X \leq N\text{-}r\} \qquad (18)$$

$$\text{for } 2 \leq \Lambda \leq N.$$

III. Summary of the stock level conditional probabilities, $a(i|AB)$.--The stock level conditional probabilities, $a(i|AB)$, are obtained by inserting expression (16) and (18) into expression (14). These probabilities are summarized as follows:

$$a(i|AB) = \frac{\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(r-2)=\nu\}P\{D(N-r+1)=S-i\}G_1(r,\nu)}{\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(r-2)=\nu\}G_1(r,\nu)}, \qquad (19)$$

$$\text{where } G_1(r,\nu) = P\{D(1) \geq S-RP-\nu\}P\{X \leq N-r\}, \qquad (19a)$$

$$\text{for } 2 \leq \Lambda \leq N \text{ and } i \leq S.$$

Stock level conditional probabilities, $a(i|A\bar{B})$.--The conditional probabilities that the beginning stock level is equal to i, given that a within-the-cycle replenishment order was placed and not received during the previous cycle, are denoted as $a(i|A\bar{B})$. These probabilities will be determined from the expression

$$a(i|A\bar{B}) = \frac{P\{A\bar{B}F_i\}}{P\{A\bar{B}\}} \qquad \text{for } i \leq S. \qquad (20)$$

I. Probability of the set $A\bar{B}F_i$.--The joint probability that a replenishment order is placed at the beginning of any of the periods during the first Λ periods, that this within-the-cycle replenishment order is not received until during the subsequent order cycle, and that the stock level at the beginning of the subsequent cycle is equal to i will be determined. A specific path in the set $A\bar{B}F_i$ occurs if the following independent conditions exist:

1. The cumulative demand at the beginning of any period during the first Λ periods of the order cycle is equal to or exceeds S-RP, causing a within-the-cycle replenishment order to be placed. This within-the-cycle replenishment order is placed at the beginning of period r if demand during the first r-2 periods is equal to ν, which is less than S-RP, and if demand during the first r-1 periods exceeds S-RP by δ items.
2. This within-the-cycle replenishment order requires a lead time greater than N-r periods.
3. Demand during the entire cycle is equal to S-i.

For specific values of r, ν, and δ, the probability of this path is

$$P\{D(r-2)=\nu\}P\{D(1)=S-RP-\nu+\delta\}P\{D(N-r+1)=RP-\delta-i\}P\{X > N-r\} \qquad (21)$$

$$\text{for } 2 \leq r \leq \Lambda \leq N,\ 0 \leq \nu \leq S-RP-1,\ 0 \leq \delta \leq RP-i,$$
$$\text{and } i \leq S.$$

If the within-the-cycle replenishment order is placed at the beginning of period r, the demand during the first r-2 periods can be any of the possible values from 0 to S-RP-1 inclusive. Since the within-the-cycle replenishment order can be placed during any of the first Λ periods except the first, r can be any of the possible values from 2 to Λ inclusive. Also, the cumulative cycle demand may exceed the minimum reorder quantity, S-RP, by δ items which are at most RP-i items. Therefore,

$$P\{A\bar{B}F_i\} = \sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} \sum_{\delta=0}^{RP-i} P\{D(1)=S-RP-\nu+\delta\}P\{D(N-r+1)=RP-\delta-i\}G_2(r,\nu), \tag{22}$$

$$\text{where } G_2(r,\nu) = P\{D(r-2)=\nu\}P\{X > N-r\}, \tag{22a}$$

for $2 \leq \Lambda \leq N$ and $i \leq S$.

II. Probability of the condition, $A\bar{B}$.--The probability that a within-the-cycle replenishment order is placed and not received during the same order cycle can be developed similarly to the $P\{AB\}$. The only difference is that the random variable X must be greater than N-r, rather than equal to or less than N-r. Therefore,

$$P\{A\bar{B}\} = \sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(r-2)=\nu\}P\{D(1) \geq S-RP-\nu\}P\{X > N-r\} \tag{23}$$

for $2 \leq \Lambda \leq N$.

III. Summary of the stock level conditional probabilities, $a(i|A\bar{B})$.--The stock level conditional probabilities, $a(i|A\bar{B})$, are obtained by inserting expressions (22) and (23) into expression (20). These probabilities are summarized as follows:

$$a(i|A\bar{B}) = \frac{\displaystyle\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} \sum_{\delta=0}^{RP-i} P\{D(1)=S-RP-\nu+\delta\}P\{D(N-r+1)=RP-\delta-i\}G_2(r,\nu)}{\displaystyle\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(1) \geq S-RP-\nu\}G_2(r,\nu)}, \tag{24}$$

$$\text{where } G_2(r,\nu) = P\{D(r-2)=\nu\}P\{X > N-r\}, \tag{24a}$$

$$\text{for } 2 \leq \Lambda \leq N \text{ and } i \leq S.$$

Stock level conditional probabilities, $a(i|A\bar{B},\bar{r})$.--The conditional probabilities that the beginning stock level is equal to i, given that a within-the-cycle replenishment order was placed during the previous cycle, was not received during that cycle, but was received at the beginning of period $\bar{r}$ during the subsequent cycle, are denoted by $a(i|A\bar{B},\bar{r})$. These probabilities can be developed similarly to $a(i|A\bar{B})$. The only difference is that the random variable X is equal to N-r+$\bar{r}$, rather than greater than N-r. Therefore,

$$a(i|A\bar{B},\bar{r}) = \frac{\sum_{r=2}^{\Lambda}\sum_{\nu=0}^{S-RP-1}\sum_{\delta=0}^{RP-i} P\{D(1)=S-RP-\nu+\delta\}P\{D(N-r+1)=RP-\delta-i\}G_3(r,\nu)}{\sum_{r=2}^{\Lambda}\sum_{\nu=0}^{S-RP-1} P\{D(1) \geq S-RP-\nu\}G_3(r,\nu)}, \tag{25}$$

$$\text{where } G_3(r,\nu) = P\{D(r-2)=\nu\}P\{X=N-r+\bar{r}\}, \tag{25a}$$

$$\text{for } 2 \leq \Lambda \leq N,\ 1 \leq \bar{r} \leq \Lambda-1, \text{ and } i \leq S.$$

Beginning Stock Level Stationary Probabilities

The beginning stock level unconditional stationary probabilities can be determined in terms of the beginning stock level conditional stationary probabilities. That is,

$$a(i) = a(i|\bar{A})P\{\bar{A}\} + a(i|AB)P\{AB\} + a(i|A\bar{B})P\{A\bar{B}\} \tag{26}$$

$$\text{for } i \leq S.$$

The probabilities, $a(i)$ are the stationary probabilities of a Markov chain. These probabilities can also be obtained by solving the following matrix equation:

$$\underline{a} = \underline{a}\, M, \tag{27}$$

$$\text{where } \underline{a} = [a(S),\ a(S-1),\ .\ .\ .,\ a(0),\ .\ .\ .], \tag{27a}$$

$$\text{and } M = \begin{bmatrix} m_{S,S} & m_{S,S-1} & \cdots & m_{S,0} & \cdots \\ m_{S-1,S} & m_{S-1,S-1} & \cdots & m_{S-1,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{0,S} & m_{0,S-1} & \cdots & m_{0,0} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{27b}$$

In the infinite matrix, M, m_{ij} is the probability that the stock level is j items at the beginning of the subsequent cycle, given that the stock level was i items at the beginning of present cycle.

Period Stock Level Stationary Probabilities

The stationary probabilities that the stock level is equal to h prior to the demand in period r are denoted by $a_r(h)$. These probabilities can be determined in terms of three period stock level conditional probabilities, $a_r(h|A)$, $a_r(h|AB)$ and $a_r(h|A\bar{B})$, in conjunction with the probability of the specific condition. The conditions; A, AB, and $A\bar{B}$; refer to the previous order cycle, being mutually exclusive and exhaustive, that is,

$$a_r(h) = [a_r(h|\bar{A})]P\{\bar{A}\} + [a_r(h|AB)]P\{AB\} + [a_r(h|A\bar{B})]P\{A\bar{B}\} \quad (28)$$
$$\text{for } 1 \leq r \leq N \text{ and } h \leq S.$$

These three period stock level conditional probabilities will be developed from N period stock level conditional probabilities, which are subject to the additional condition of a specific replenishment lead time, in conjunction with the probability of this specific replenishment lead time. Each of the sets of N conditions are mutually exclusive and exhaustive. That is,

$$a_r(h|\bar{A}) = \sum_{x=0}^{N-1} [a_r(h|\bar{A},x)]P\{X=x\} \quad \text{for } 1 \leq r \leq N \text{ and } h \leq S, \quad (29)$$

$$a_r(h|AB) = \sum_{x=0}^{N-1} [a_r(h|AB,x)]P\{X=x\} \text{ for } 1 \leq r \leq N \text{ and } h \leq S, \quad (30)$$

and $$a_r(h|A\bar{B}) = \sum_{x=0}^{N-1} [a_r(h|A\bar{B},x)]P\{X=x\} \text{ for } 1 \leq r \leq N \text{ and } h \leq S. \quad (31)$$

For convenience the more difficult expression $a_r(h|A\bar{B},x)$ will be developed first. From this expression and by making specific observations, the expressions $a_r(h|\bar{A},x)$ and $a_r(h|AB,x)$ will follow without a formal argument. In the development of these period stock level conditional probabilities; $a_r(h|A\bar{B},x)$, $a_r(h|\bar{A},x)$, and $a_r(h|AB,x)$; it is convenient to define the following sets:

1. Set C: Contains all of the paths in which a replenishment order is placed within the minimum of Λ or r periods.

2. Set $\bar{C}$: Contains all of the paths in which a replenishment order is not placed within the minimum of Λ or r periods.
3. Set D: Contains all of the paths in which the within-the-cycle replenishment order is received within the first r periods in the same order cycle in which the replenishment order is placed.
4. Set $\bar{D}$: Contains all of the paths in which the within-the-cycle replenishment order is not received within the first r periods in the same order cycle in which the replenishment is placed.
5. Set G_t: Contains all of the paths in which the beginning stock on order is t items.
6. Set H_h: Contains all of the paths in which the period stock level is h items.

Development of the period stock level conditional probabilities, $a_r(h|A\bar{B},x)$.--The conditional probabilities that the stock level is equal to h at the beginning of period r, $a_r(h|A\bar{B},x)$, given that a within-the-cycle replenishment order was placed during the previous cycle and was not received during that order cycle and given that the replenishment lead time for the beginning cycle order is x, are denoted by $a_r(h|A\bar{B},x)$.

These period stock level conditional probabilities, $a_r(h|A\bar{B},x)$, will be developed from Λ-1 period stock level conditional probabilities, which are subject to the additional condition of receiving the carry-over replenishment order at the beginning of a particular period, $\bar{r}$. These Λ-1 conditions are mutually exclusive and exhaustive, and the random variables X and $\bar{R}$ are independent. Therefore

$$a_r(h|A\bar{B},x) = \sum_{\bar{r}=1}^{\Lambda-1} [a_r(h|A\bar{B},x,\bar{r})]P\{\bar{R}=\bar{r}|A\bar{B}\} \tag{32}$$

$$\text{for } 1 \leq r \leq N,\ h \leq S, \text{ and } 0 \leq x \leq N\text{-}1.$$

The following expressions will be useful in developing $a_r(h|A\bar{B},x,\bar{r})$: the beginning stock level conditional probabilities, $a(i|A\bar{B},\bar{r})$ (previously developed); the stock-on-order conditional stationary probabilities, $u(t|A\bar{B})$; and the beginning stock-level and stock-on-order conditional probabilities, $b(z|A\bar{B},r)$; and the receipt of carry-over replenishment order conditional probabilities, $P\{\bar{R}=\bar{r}|A\bar{B}\}$.

Beginning stock-on-order conditional stationary probabilities.--The conditional probabilities that the stock on order is equal to t, given that a within-the-cycle replenishment order was placed during the previous cycle, was not received during that cycle, but was received at the beginning of period $\bar{r}$ of the present cycle, are denoted by $u(t|A\bar{B},\bar{r})$. These probabilities will be determined from the expression

$$u(t|A\bar{B},\bar{r}) = \frac{P\{A\bar{B},\bar{r},G_t\}}{P\{A\bar{B},\bar{r}\}} \qquad \text{for } t \geq S\text{-}RP. \tag{33}$$

I. Probability of the set $A\bar{B},\bar{r},G_t$.--The probability that a replenishment order is placed at the beginning of any of the periods during the first Λ periods of the cycle, that this within-the-cycle replenishment order is not received during the same order cycle, but is received at the beginning of period $\bar{r}$ of the subsequent cycle, and that this replenishment order is for t items, will be determined. A specific path in the set $A\bar{B},\bar{r},G_t$ occurs if the following independent conditions exist:

1. The cumulative demand at the beginning of any period during the first Λ periods of the order cycle is equal to t, which is equal to or greater than S-RP, causing a within-the-cycle replenishment order to be placed. This within-the-cycle order is placed for t items at the beginning of period r if demand during the first r-2 periods is equal to ν, which is less than S-RP, and if demand during the first r-1 periods is equal to t items, for t is equal to or greater than S-RP.
2. This within-the-cycle replenishment order requires a lead time greater than N-r periods, but equal to $N-r+\bar{r}$ periods.

For specific values of r and ν, the probability of this path is

$$P\{D(r-2)=\nu\}P\{D(1)=t-\nu\}P\{X=N-r+\bar{r}\} \qquad (34)$$

for $2 \leq r \leq \Lambda$, $S\text{-}RP \leq t$, $0 \leq \nu \leq S\text{-}RP\text{-}1$, $0 \leq x \leq N\text{-}1$, and $1 \leq \bar{r} \leq \Lambda\text{-}1$.

Since the within-the-cycle replenishment order can be placed at the beginning of any of the periods during the first Λ periods, except the first, r can be any of the possible values from 2 to Λ inclusive. If the within-the-cycle replenishment order is placed at the beginning of period r, demand during the r-2 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore,

$$P\{A\bar{B},\bar{r},G_t\} = \sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(r-2)=\nu\}P\{D(1)=t-\nu\}P\{X=N-r+r\} \qquad (35)$$

for $S\text{-}RP \leq t$, $0 \leq x \leq N\text{-}1$, and $1 \leq \bar{r} \leq \Lambda\text{-}1$.

II. Probability of the condition, $A\bar{B},\bar{r}$.--The probability that a replenishment order is placed, not received during the same order cycle, but received at the beginning of period $\bar{r}$ during the subsequent cycle can be determined similarly to $P\{A\bar{B}\}$. The only difference is that the random variable X is equal to $N-r+\bar{r}$, rather than greater than $N-r$. Therefore,

$$P\{A\bar{B},\bar{r}\} = \sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(r-2)=\nu\}P\{D(1) \geq S-RP-\nu\}P\{X=N-r+\bar{r}\} \quad (36)$$

$$\text{for } 0 \leq x \leq N-1 \text{ and } 2 \leq \Lambda \leq N.$$

III. Summary of stock-on-order conditional probabilities.--The stock-on-order conditional probabilities, $u(t|A\bar{B},\bar{r})$, are obtained by inserting expressions (35) and (36) into expression (33). These probabilities are summarized as follows:

$$u(t|A\bar{B},\bar{r}) = \frac{\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(1)=t-\nu\}G_3(r,\nu)}{\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(1) \geq S-RP-\nu\}G_3(r,\nu)}, \quad (37)$$

$$\text{where } G_3(r,\nu) = P\{D(r-2)=\nu\}P\{X=N-r+\bar{r}\}, \quad (37a)$$

$$\text{for } 0 \leq x \leq N-1 \text{ and } 2 \leq \Lambda \leq N.$$

IV. Beginning stock level and stock-on-order conditional stationary probabilities $b(z|A\bar{B},\bar{r})$.--The condtional stationary probabilities that the sum of stock level and stock on order is equal to z, given that a within-the-cycle replenishment order was placed during the previous cycle, was not received during that order cycle, but was received at the beginninf of period $\bar{r}$ of the present cycle, are denoted by $b(z,A\bar{B},\bar{r})$. The values which z assumes are z=S, S-1, ... This development is similar to the development of $a(i|A\bar{B},\bar{r})$, except that the demand during the N-r+1 periods must be equal to S-z if the sum of stock level and stock on order at the beginning of the subsequent cycle equals z. Also, the minimum order quantity, S-RP, may be exceeded by any quantity. The resulting expression is summarized as follows:

$$b(z|A\bar{B},\bar{r}) = \frac{\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(1) \geq S-RP-\nu\}P\{D(N-r+1)=S-z\}G_3(r,\nu)}{\sum_{r=2}^{\Lambda} \sum_{\nu=0}^{S-RP-1} P\{D(1) \geq S-RP-\nu\}G_3(r,\nu)}, \quad (38)$$

$$\text{where } G_3(\nu,r) = P\{D(r-2)=\nu\}P\{X=N-r+\bar{r}\}, \quad (38a)$$

$$\text{for } 2 \leq \Lambda \leq N \text{ and } z \leq S.$$

V. <u>Receipt of carry-over replenishment order conditional probabilities</u>.--The conditional probabilities that a carry-over order is received at the beginning of period r, $P\{\bar{R}=\bar{r}|A\bar{B}\}$, given that a within-the-cycle replenishment order was placed during the previous cycle and not received during that order cycle, are denoted by $P\{\bar{R}=\bar{r}|A\bar{B}\}$. The random variable, $\bar{R}$, corresponds to the period in which the carry-over replenishment order is received; and $\bar{r}$ = 1, 2, . . ., Λ-1 are the values which this random variable assumes. These conditional probabilities for the receipt of the carry-over replenishment order will be used as shown in expression (32) in the development of the period stock level conditional probabilities, $a_r(h|A\overline{B},x)$.

As an intermediate step in obtaining $P\{\overline{R}=\bar{r}|A\overline{B}\}$, it is convenient to determine the joint probabilities that a replenishment order is carried over and that this order is received at the beginning of period $\bar{r}$ in the subsequent cycle, which are denoted as $P\{\overline{R}=\bar{r},A\overline{B}\}$. Also, the probabilities that the within-the-cycle replenishment order is placed at the beginning of period w, $P\{W=w\}$, will be determined. The random variable W corresponds to the period in which the within-the-cycle replenishment order is placed, and $w = 2, 3, \ldots, \Lambda$ are the values which this random variable assumes.

A within-the-cycle replenishment order is placed at the beginning of period w if demand during the first w-2 periods is equal to ν, for $0 \leq \nu \leq$ S-RP-1, and if demand during the next period is equal to or greater than S-RP-ν. The probability of this specific path is

$$P\{D(w-2)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}\nu\} \qquad \text{for } 0 \leq \nu \leq S\text{-}RP\text{-}1 \text{ and } 2 \leq w \leq \Lambda. \tag{39}$$

If the within-the-cycle replenishment is placed at the beginning of period w, demand during the first w-2 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore,

$$P\{W=w\} = \sum_{\nu=0}^{S\text{-}RP\text{-}1} P\{D(w-2)=\nu\}P\{D(1) \geq S\text{-}RP\text{-}\nu\} \qquad \text{for } 2 \leq w \leq \Lambda. \tag{40}$$

A specific path in which a within-the-cycle replenishment order is carried over into the subsequent cycle and is received at the beginning of period $\bar{r}$ in this subsequent cycle occurs if the within-the-cycle replenishment order, W, is placed at the beginning of period w, for $2 \leq w \leq \Lambda$, and if the lead time for this replenishment order is equal to $N-w+\bar{r}$. The probability of this specific path is

$$P\{W=w\}P\{X=N-w+\bar{r}\} \quad \text{for } 2 \leq w \leq \Lambda \leq N \text{ and } 1 \leq \bar{r} \leq \Lambda-1. \tag{41}$$

Since the random variable W can be any of the possible values from 2 to Λ inclusive,

$$P\{\bar{R}=\bar{r},A\bar{B}\} = \sum_{w=2}^{\Lambda} P\{W=w\}P\{X=N-w+\bar{r}\} \quad \text{for } 1 \leq \bar{r} \leq \Lambda-1. \tag{42}$$

The conditional probability that a carry-over order is received at the beginning of a particular period $\bar{r}$, given that a within-the-cycle replenishment order is placed and not received during the previous order cycle, can be obtained by inserting expressions (42) and (23) into the following expression:

$$P\{\bar{R}=\bar{r}|A\bar{B}\} = \frac{P\{\bar{R}=\bar{r},A\bar{B}\}}{P\{A\bar{B}\}} \qquad \text{for } 1 \leq \bar{r} \leq \Lambda-1. \tag{43}$$

VI. <u>Development of the period stock level conditional probabilities, $a_r(h|A\bar{B},x,\bar{r})$</u>.--The conditional probabilities that the stock level is equal to h prior to the demand in period r, given that a within-the-cycle replenishment order was placed during the previous cycle, was not received during that cycle, but was received at the beginning of period $\bar{r}$

of the present cycle, and given that the replenishment lead time for the beginning cycle order is x periods, are denoted as $a_r(h|A\bar{B},x,\bar{r})$. In the development of $a_r(h|A\bar{B},x,\bar{r})$, consider the cases corresponding to the four mutually exclusive and exhaustive sets: $r < \bar{r}$ and $r < x+1$, $r \geq \bar{r}$ and $r < x+1$, $r < \bar{r}$ and $r \geq x+1$, $r \geq \bar{r}$ and $r \geq x+1$.

A. Case I: $r < \bar{r}$ and $r < x+1$.--In this case neither the beginning cycle replenishment order nor the carry-over replenishment order have been received. Consequently, the stock level at the beginning of period 1 is equal to the beginning stock level. Therefore,

$$a_1(h|A\bar{B},x,\bar{r}) = a(h|A\bar{B},\bar{r}) \qquad \text{for } 1 < \bar{r} \leq \Lambda\text{-}1 \quad \text{and} \quad 1 < x+1 \leq N. \tag{44}$$

For $r > 1$, consider the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $h \leq RP$ and $RP < h \leq S$.

1. Sub-case I: $h \leq RP$.--The stock level is equal to h, for $h \leq RP$, prior to the demand in period r, only by the paths contained in the set

$$\bar{C}H_h + C\bar{D}H_h + CDH_h \,. \tag{45}$$

Since this set is the union of mutually exclusive sets,

$$a_r(h|A\bar{B},x,\bar{r}) = P\{\bar{C}H_h\}^* + P\{C\bar{D}H_h\} + P\{CDH_h\} \tag{46}$$

$$\text{for } h \leq RP,\ 2 \leq r \leq N,\ 0 \leq x \leq N\text{-}1,\ \Lambda \leq N,$$
$$2 \leq r < \bar{r} \leq \Lambda\text{-}1, \text{ and } 2 \leq r < x+1 \leq N.$$

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*For convenience of notation, the conditions; $A\bar{B},x,\bar{r}$; will be omitted from $P\{\bar{C}H_h\}$ and similar probabilities throughout this section.

a. Probability of the paths in the set $\overline{CH}_h$.--In this development, consider the sub-cases corresponding to the sub-sets obtained by dichotomizing this set into the following mutually exclusive and exhaustive sub-sets: $2 \leq r \leq \Lambda \leq N-1$ and $2 \leq \Lambda < r \leq N$.

(1) Sub-case I: $r \leq \Lambda$.--The probability that a replenishment order is not placed within the first r periods and that the stock level is equal to h, for $h \leq RP$, prior to the demand in period r, for $2 \leq r \leq \Lambda$, will be determined. A specific path in the set $\overline{CH}_h$ occurs if the beginning stock level is equal to i, for $h \leq i \leq h+(S\text{-}RP\text{-}1)$, and if demand during the first r-1 periods is equal to i-h. The probability of this specific path is

$$a(i|A\overline{B},\bar{r})P\{D(r-1)=i-h\} \quad \text{for } h \leq i \leq h+(S\text{-}RP\text{-}1), \tag{47}$$
$$2 \leq r \leq \Lambda \leq N,\ h \leq RP,\ 2 \leq r < \bar{r} \leq \Lambda-1,$$
$$\text{and } 2 \leq r < x+1 \leq N.$$

Since the beginning stock level can be any of the possible values from h to h+(S-RP-1) inclusive,

$$P\{\overline{CH}_h\} = \sum_{i=h}^{h+(S\text{-}RP\text{-}1)} a(i|A\overline{B},\bar{r})P\{D(r-1)=i-h\} \tag{48}$$
$$\text{for } 2 \leq r \leq \Lambda \leq N,\ h \leq r < \bar{r} \leq \Lambda-1,$$
$$\text{and } 2 \leq r < x+1 \leq N.$$

(2) Sub-case II: $\Lambda < r$.--The probability that a replenishment order is not placed within the order cycle and that the stock level is equal to h, for $h \leq RP$, prior to the demand in period r, $\Lambda < r$,

will be determined. A specific path in the set $\bar{C}H_h$ occurs if the following three conditions exist:

1. The beginning stock level is equal to i, for $h \leq i \leq S$.
2. Demand during the first Λ-1 periods is equal to ν, for $0 \leq \nu \leq S\text{-}RP\text{-}1$.
3. Demand during the next r-Λ+1 periods is equal to i-ν-h.

For specific values of i and ν, the probability of this specific path is

$$a(i|A\bar{B},\bar{r})P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda)=i-\nu-h\} \tag{49}$$

$$\text{for } h \leq i \leq S,\ 2 \leq \Lambda < r \leq N,\ 0 \leq \nu \leq S\text{-}RP\text{-}1,$$

$$h \leq RP,\ 2 \leq r < \bar{r} \leq \Lambda\text{-}1, \text{ and } 2 \leq r < x+1 \leq N.$$

The beginning stock level can be any of the possible values from h to S inclusive. Demand during the first Λ-1 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore,

$$P\{\bar{C}H_h\} = \sum_{i=h}^{S} \sum_{\nu=0}^{S-RP-1} a(i|A\bar{B},\bar{r})P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda)=i-\nu-h\} \tag{50}$$

$$\text{for } 2 \leq \Lambda < r \leq N,\ h \leq RP,\ 2 \leq r < \bar{r} \leq \Lambda\text{-}1,$$

$$\text{and } 2 \leq r < x+1 \leq N.$$

b. <u>Probability of the paths in the set $C\bar{D}H_h$</u>.--The probability that a within-the-cycle replenishment order is placed at the beginning of any of the periods during the minimum of the first Λ or r periods, that this replenishment order is not received, and that the stock level at the beginning of period r is equal to h will be determined.

A specific path in the set $C\bar{D}H_h$ occurs if the following four conditions exist:

1. The beginning stock level is equal to i.
2. The cumulative demand at the beginning of any period during the minimum of the first Λ or r periods of the order cycle is equal to or greater than S-RP, causing a within-the-cycle replenishment order to be placed. This within-the-cycle replenishment order is placed at the beginning of period r' if the demand during the first r'-2 periods is equal to ν, which is less than S-RP, and if the demand during the first r'-1 periods exceeds S-RP by δ items.
3. This within-the-cycle replenishment order requires a lead time greater than r-r' periods.
4. Demand during the first r-1 periods is equal to i-h.

For specific values of i, r, ν, and δ, the probability of this path is

$$a(i|\bar{A}\bar{B},r)P\{D(r'-2)=\nu\}P\{X > r-r'\}g_1(i,r',\nu), \tag{51}$$

where $g_1(i,r',\nu) = P\{D(1)=S\text{-}RP-\nu+\delta\}P\{D(r-r')=i-(S\text{-}RP+\delta)-h\}$, (51a)

for $h+S\text{-}RP \le i \le S$, $0 \le \delta \le (i-h)-(S\text{-}RP)$, $2 \le r' \le r < \Lambda-1 \le N-1$,

$0 \le \nu \le S\text{-}RP-1$, $2 \le r < \bar{r} \le \Lambda-1$, and $2 \le r < x+1 \le N$.

If the within-the-cycle replenishment order is placed at the beginning of period r', the demand during the r'-2 periods can be any of the possible values from 0 to S-RP-1 inclusive. The beginning stock level, i, must be greater than h by at least S-RP or no within-the-cycle replenishment order is placed; therefore, the index of i ranges from h+ S-RP to S. The within-the-cycle replenishment order may be placed at the beginning of any period in either the first Λ or r periods, whichever is smaller except the first; hence, r' can be any of the possible values from 2 to the minimum of either Λ or r inclusive. Also the cumulative cycle demand at the end of period r'-1 may exceed the minimum reorder quantity S-RP by δ items which are at most (i-h) - (S-RP) items. Therefore,

$$P\{C\bar{D}H_h\} = \sum_{i=h+S-RP}^{S} \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} a(i|A\bar{B},\bar{r})P\{D(r'-2)=\nu\}P\{X > r-r'\}G_1(i,r',\nu), \tag{52}$$

$$\text{where } G_1(i,r',\nu) = \sum_{\delta=0}^{(i-h)-(S-RP)} P\{D(1)=S-RP-\nu+\delta\}P\{D(r-r')=i-(S-RP+\delta)-h\},$$

$$\text{for } 2 \leq r < \Lambda-1 \leq N-1,\ h \leq RP,\ 2 \leq r < \bar{r} \leq \Lambda-1, \text{ and } 2 \leq r < x+1 \leq N. \tag{52a}$$

c. Probability of the paths in the set CDH_h.--The probability that a within-the-cycle replenishment order is placed at the beginning of any of the periods during the minimum of the first Λ or r periods, that this replenishment order is received at or before the beginning of period r, and that the stock level at the beginning of period r is equal to h will be determined. A specific path in the set CDH_h occurs if the following four conditions exist:

1. The beginning stock level is equal to i.
2. The cumulative demand at the beginning of any period during the minimum of Λ or r periods of the order cycle is equal

to or greater than S-RP, causing a within-the-cycle replenishment order to be placed. This within-the-cycle replenishment order is placed at the beginning of period r' if demand during the first r'-2 periods is equal to ν, which is less than S-RP, and if the demand during the first r'-1 periods is equal to or greater than S-RP.

3. This within-the-cycle replenishment order is received within r'-r periods.
4. Demand during the r - r' periods after the replenishment order is placed is equal to i-h.

For specific values of i, r, and ν, the probability of this path is

$$a(i|\overline{AB},\bar{r})P\{D(r'-2)=\nu\}P\{X \leq r-r'\}G_2(i,r',\nu), \tag{53}$$

$$\text{where } G_2(i,r',\nu) = P\{D(1) \geq S\text{-}RP-\nu\}P\{D(r-r')=i-h\}, \tag{53a}$$

$$\text{for } h \leq i \leq S,\ 2 \leq r' \leq r < \Lambda-1 \leq N-1,\ 0 \leq \nu \leq S\text{-}RP-1,$$

$$2 \leq r < \bar{r} \leq \Lambda-1, \text{ and } 2 \leq r < x+1 \leq N.$$

If the within-the-cycle replenishment order is placed at the beginning of period r', the demand during the r'-2 periods can be any of the possible values from 0 to S-RP-1 inclusive. The beginning stock level, i, must be equal to or greater than h; therefore, the beginning stock level can be any of the values from h to S inclusive. The within-the-cycle replenishment order may be placed at the beginning of any period in either the first Λ or r periods, whichever is the smaller, except the first period; hence r' can be any of the possible values from 2 to the minimum of either Λ or r inclusive. Therefore,

$$P\{CDH_h\} = \sum_{i=h}^{S} \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} a(i|A\bar{B},\bar{r})P\{D(r'-2)=\nu\}P\{X \leq r-r'\}G_2(i,r',\nu), \tag{54}$$

$$\text{where } G_2(i,r',\nu) = P\{D(1) \geq S-RP-\nu\}P\{D(r-r')=i-h\}, \tag{54a}$$

$$\text{for } h \leq S,\ 2 \leq r < \bar{r} \leq \Lambda-1, \text{ and } 2 \leq r < x+1 \leq N.$$

2. Sub-case II: $RP < h \leq S$.--The stock level is equal to h, for $RP < h \leq S$, prior to the demand in period r by the paths contained in the set

$$\bar{C}H_h + CDH_h. \tag{55}$$

Since this set is the union of mutually exclusive sets,

$$a_r(h|A\bar{B},x,\bar{r}) = P\{\bar{C}H_h\} + P\{CDH_h\} \tag{56}$$

$$\text{for } RP < h \leq S,\ 2 \leq r \leq N,\ 0 \leq x \leq N-1,$$

$$1 \leq \bar{r} \leq \Lambda-1 \leq N-1,\ 2 \leq r < \bar{r} \leq \Lambda-1,$$

$$\text{and } 2 \leq r < x+1.$$

a. Probability of the paths in the set $\bar{C}H_h$.--The probability that a replenishment order is not placed within the minimum of Λ or r periods and that the stock level is equal to h, for $RP < h \leq S$, prior to the demand in period r will be determined. A specific path in the set $\bar{C}H_h$ occurs if the beginning stock level is equal to i, for $h \leq i \leq S$, and demand during the r-1 periods is equal to i-h. The probability of this specific path is

$$a(i|A\bar{B},\bar{r})P\{D(r-1)=i-h\} \tag{57}$$

for $RP < h \leq i \leq S$, $2 \leq r < \bar{r} \leq \Lambda-1$, and

$2 \leq r < x+1 \leq N$.

The beginning stock level, i, can be any one of the possible values from h to S inclusive. Therefore,

$$P\{\overline{C}H_h\} = \sum_{i=h}^{S} a(i|A\bar{B},\bar{r})P\{D(r-1)=i-h\} \tag{58}$$

for $RP < h \leq S$, $2 \leq r < \bar{r} \leq \Lambda-1$, and $2 \leq r < x+1 \leq N$.

b. Probability of the paths in the set CDH_h.--The probability that a replenishment order is placed at the beginning of any of the periods during the minimum of the first Λ or r periods, that this replenishment order is received at or before the beginning of period r, and that the stock level is equal to h, for $RP < h \leq S$, prior to the demand in period r was determined as expression (54). The only restriction for expression (54) upon h is that $h \leq S$.

3. Summary of Case I: $r < \bar{r}$ and $r < x+1$.--Case I will be summarized for r=1 and for $r > 1$. Expression (44) is rewritten for r=1. The period stock level probabilities corresponding to Sub-case I for $r > 1$ are obtained by combining and simplifying expressions (48), (50), (52), and (54). The period stock level conditional probabilities corresponding to Sub-case II for $r > 1$ are obtained by combining and simplifying expressions (54) and (58). Therefore,

$$a_r(h|A\bar{B},x,\bar{r}) = \begin{cases} a(h|A\bar{B},\bar{r}) \qquad \text{for } h \le S,\ r=1,\ 1 < \bar{r} \le \Lambda-1 \le N-1 \text{ and } 1 < x+1 \le N; \\[2ex] H_1 + \displaystyle\sum_{r'=2}^{\min(\Lambda,r)} \Bigg[\sum_{\nu=0}^{S-RP-1} P\{D(r'-2)=\nu\} \sum_{i=h+S-RP}^{S} a(i|A\bar{B},\bar{r})P\{X > r-r'\}G_1(i,r',\nu) \\ \qquad + \displaystyle\sum_{i=h}^{S} a(i|A\bar{B},\bar{r})P\{X \le r-r'\}G_2(i,r',\nu)\Bigg] \\ \qquad \text{for } h \le RP,\ 2 \le r < \bar{r} \le \Lambda-1 \le N-1, \text{ and } 2 \le r < x+1 \le N; \\[2ex] \displaystyle\sum_{i=h}^{S} a(i|A\bar{B},\bar{r})\{D(r-1)=i-h\} \\ \qquad + \displaystyle\sum_{i=h}^{S} \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} a(i|A\bar{B},\bar{r})P\{D(r'-2)=\nu\}P\{X \le r-r'\}G_2(i,r',\nu) \\ \qquad \text{for } RP < h \le S,\ 2 \le r < \bar{r} \le \Lambda \le N-1, \text{ and } 2 \le r < x+1 \le N; \end{cases} \tag{59}$$

where $$G_1(i,r',\nu) = \sum_{\delta=0}^{(i-h)-(S-RP)} P\{D(1)=S-RP-\nu+\delta\}P\{D(r-r')=i-(S-RP+\delta)-h\}, \tag{59a}$$

$$G_2(i,r',\nu) = P\{D(1) \geq S-RP-\nu\}P\{D(r-r')=i-h\}, \tag{59b}$$

and $$H_1 = \begin{cases} \displaystyle\sum_{i=h}^{S} \sum_{\nu=0}^{S-RP-1} a(i|A\bar{B},\bar{r})P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda+1)=i-\nu-h\} \\ \qquad \text{for } 2 \leq \Lambda < r \leq N, \\ \displaystyle\sum_{i=h}^{h+(S-RP-1)} a(i|A\bar{B},\bar{r})P\{D(r-1)=i-h\} \\ \qquad \text{for } 2 \leq r \leq \Lambda \leq N. \end{cases} \tag{59c}$$

B. Case II: $r \geq \bar{r}$ and $r < x+1$.--In this case the carry-over replenishment order has been received, with the beginning cycle replenishment order outstanding. Therefore, the stock available during the first r periods is the sum of beginning stock level and stock on order and is denoted by z. The conditional probability of z items, given the condition $A\bar{B},\bar{r}$, is denoted by $b(z|A\bar{B},\bar{r})$. The development for Case II is identical with the development of Case I except that the expressions involving $a(i|A\bar{B},\bar{r})$, and i are replaced by $b(z|A\bar{B},\bar{r})$ and z, respectively. The resulting expression for $a_r(h|A\bar{B},x,\bar{r})$ is summarized as follows:

$$
a_r(h|A\bar{B},x,\bar{r}) = \begin{cases}
b(h|A\bar{B},\bar{r}) \qquad \text{for } h \le S,\ r=1,\ 1=\bar{r} \le \Lambda-1 \le N-1, \text{ and } 1 < x+1 \le N; \\[2ex]
H_2 + \displaystyle\sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} P\{D(r'-2)=\nu\}\Bigg[\sum_{z=h+S-RP}^{S} b(z|A\bar{B},\bar{r})P\{X > r-r'\}G_3(z,r',\nu) \\
\qquad + \displaystyle\sum_{z=h}^{S} b(z|A\bar{B},\bar{r})P\{X \le r-r'\}G_4(z,r',\nu)\Bigg] \\
\qquad\qquad \text{for } h \le RP,\ 2 \le \bar{r} \le r \le N-1, \text{ and } 2 \le r < x+1 \le N; \\[2ex]
\displaystyle\sum_{z=h}^{S} b(z|A\bar{B},\bar{r})P\{D(r-1)=z-h\} \\
\qquad + \displaystyle\sum_{z=h}^{S} \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} b(z|A\bar{B},\bar{r})P\{D(r'-2)=\nu\}P\{X \le r-r'\}G_4(z,r',\nu) \\
\qquad\qquad \text{for } RP < h \le S,\ 2 \le \bar{r} \le r \le N-1, \text{ and } 2 \le r < x+1 \le N;
\end{cases} \tag{60}
$$

$$\text{where } G_3(z,r',\nu) = \sum_{\delta=0}^{(z-h)-(S-RP)} P\{D(1)=S-RP-\nu+\delta\}P\{D(r-r')=z-(S-RP+\delta)-h\}, \tag{60a}$$

$$G_4(z,r',\nu) = P\{D(1) \geq S-RP-\nu\}P\{D(r-r')=z-h\}, \tag{60b}$$

$$\text{and } H_2 = \begin{cases} \sum_{z=h}^{S} \sum_{\nu=0}^{S-RP-1} b(z|A\bar{B},\bar{r})P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda+1)=z-\nu-h\} & (60c) \\ \qquad \text{for } 2 \leq \Lambda < r \leq N, \\ \sum_{z=h}^{h+(S-RP-1)} b(z|A\bar{B},\bar{r})P\{D(r-1)=z-h\} \\ \qquad \text{for } 2 \leq r \leq \Lambda \leq N. \end{cases}$$

C. <u>Case III: $r < \bar{r}$ and $r \geq x+1$</u>.--In this case, the beginning cycle replenishment order has been received, with the carry-over replenishment order outstanding. Therefore, the stock available during the first r periods is the difference between the order level, S, and the carry-over replenishment order, t. The probability that the carry-over replenishment order is for t items is denoted by u(t). If $t = S-\alpha$, there are α items available during the first r periods; and the conditional probability of α items available during the first r periods is denoted by $u(S-\alpha|A\bar{B},\bar{r})$. The development for Case III is identical with the development of Case I except that the expressions involving $a(i|A\bar{B},\bar{r})$ and i are replaced by $u(S-\alpha|A\bar{B},\bar{r})$ and α, respectively. The resulting expression for $a_r(h|A\bar{B},x,\bar{r})$ is summarized as follows:

$$
a_r(h|A\bar{B},x,\bar{r}) = \begin{cases}
u(S-h|A\bar{B},\bar{r}) \qquad \text{for } h \le S,\ r=1,\ 1 < \bar{r} \le \Lambda-1 \le N-1, \text{ and } x=0; \\[2ex]
H_3 + \sum\limits_{r'=2}^{\min(\Lambda,r)} \sum\limits_{\nu=0}^{S-RP-1} P\{D(r'-2)=\nu\}\Big[\sum\limits_{\alpha=h+S-RP}^{S} u(S-\alpha|A\bar{B},\bar{r})P\{X > r-r'\}G_5(\alpha,r',\nu) \\
\qquad + \sum\limits_{\alpha=h}^{S} u(S-\alpha|A\bar{B},\bar{r})P\{X \le r-r'\}G_6(\alpha,r',\nu)\Big] \\
\qquad\qquad \text{for } h \le RP,\ 2 \le r < \bar{r} \le \Lambda-1 \le N-1, \text{ and } 2 \le x+1 \le r \le N; \\[2ex]
\sum\limits_{\alpha=h}^{S} u(S-\alpha|A\bar{B},\bar{r})P\{D(r-1)=\alpha-h\} \\
\qquad + \sum\limits_{\alpha=h}^{S} \sum\limits_{r'=2}^{\min(\Lambda,r)} \sum\limits_{\nu=0}^{S-RP-1} u(S-\alpha|A\bar{B},\bar{r})P\{D(r'-2)=\nu\}P\{X \le r-r'\}G_6(\alpha,r',\nu) \\
\qquad\qquad \text{for } RP < h \le S,\ 2 \le r < \bar{r} \le \Lambda \le N-1, \text{ and } 2 \le x+1 \le r \le N;
\end{cases} \tag{61}
$$

$$\text{where } G_5(\alpha,r',\nu) = \sum_{\delta=0}^{(\alpha-h)-(S-RP)} P\{D(1)=S-RP-\nu+\delta\}P\{D(r-r')=\alpha-(S-RP+\delta)-h\}, \tag{61a}$$

$$G_6(\alpha,r',\nu) = P\{D(1) \geq S-RP-\nu\}P\{D(r-r')=\alpha-h\}, \tag{61b}$$

$$\text{and } H_3 = \begin{cases} \displaystyle\sum_{\alpha=h}^{S} \sum_{\nu=0}^{S-RP-1} u(S-\alpha|A\bar{B},\bar{r})P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda+1)=\alpha-\nu-h\} & (61c) \\ \qquad\qquad \text{for } 2 \leq \Lambda < r \leq N, \\ \displaystyle\sum_{\alpha=h}^{h+(S-RP-1)} u(S-\alpha|A\bar{B},\bar{r})P\{D(r-1)=\alpha-h\} \\ \qquad\qquad \text{for } 2 \leq r \leq \Lambda \leq N. \end{cases}$$

D. Case IV: $r \geq \bar{r}$ and $r \geq X+1$.--In this case, both the beginning cycle replenishment order and the carry-over replenishment order have been received. Therefore, the stock available during the first r periods is equal to S, with unit probability. The development for Case IV is identical with the development of Case I except that the stock upon which demand is made is S items with unit probability rather than in Case I in which the stock upon which demand was made is i with probability $a(i|A\bar{B},\bar{r})$. The resulting expression for $a_r(h|A\bar{B},x,\bar{r})$ is summarized as follows:

$$a_r(h|A\bar{B},x,\bar{r}) = \begin{cases} \delta_{hS} & \text{for } r=1=\bar{r}=x+1; \qquad (62) \\[2ex] H_4 + \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} P\{D(r'-2)=\nu\} \quad P\{X > r-r'\}G_7(r',\nu) \\ \qquad + P\{X \le r-r'\}G_8(r',\nu) \\ \qquad\qquad \text{for } h \le RP,\ 1 \le \bar{r} \le r \le N,\ r \ge 2, \\ \qquad\qquad\qquad \text{and } 1 \le x+1 \le r \le N; \\[2ex] P\{D(r-1)=S-h\} \\ \qquad + \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} P\{D(r'-2)=\nu\}P\{X \le r-r'\}G_8(r',\nu) \\ \qquad\qquad \text{for } RP < h \le S,\ 1 \le \bar{r} \le r \le N,\ r \ge 2, \\ \qquad\qquad\qquad \text{and } 1 \le x+1 \le r \le N; \end{cases}$$

$$\text{where } G_7(r',\nu) = \sum_{\delta=0}^{RP-h} P\{D(1)=S-RP-\nu+\delta\}P\{D(r-r')=RP-\delta-h\}, \qquad (62a)$$

$$G_8(r',\nu) = P\{D(1) \ge S-RP-\nu\}P\{D(r-r')=S-h\}, \qquad (62b)$$

$$\text{and } H_4 = \begin{cases} \sum_{\nu=0}^{S-RP-1} P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda+1)=S-\nu-h\} \qquad (62c) \\ \qquad\qquad \text{for } 2 \le \Lambda < r \le N, \\ 0 \qquad\qquad \text{for } 2 \le r \le \Lambda \le N. \end{cases}$$

<u>Development of the period stock level conditional probabilities, $a_r(h|\bar{A},x)$</u>.--The conditional probabilities that the stock level is equal to h prior to the demand in period r, $a_r(h|\bar{A},x)$, given that no within-the-cycle replenishment order was placed during the previous cycle and that the replenishment lead time for the beginning cycle replenishment order is x, are denoted by $a_r(h|\bar{A},x)$. These probabilities will be determined in terms of the beginning stock level conditional stationary probabilities, $a(i|\bar{A})$. Consider the cases corresponding to the two mutually exclusive and exhaustive sets: $r < x+1$ and $r \geq x+1$.

For Case I with $r < x+1$, the beginning cycle replenishment order has not been received. Also since the set $\bar{A}$ does not result in a replenishment order outstanding, any reference to the period in which a within-the-cycle replenishment order is received during the subsequent cycle is not relevant in the development of $a_r(h|\bar{A},x)$. Consequently, this development is similar to the development of $a_r(h|A\bar{B},x,\bar{r})$ which was summarized in expression (59) for $r < \bar{r}$ and $r < x+1$ except that the expression $a(i|A\bar{B},\bar{r})$ is replaced by $a(i|\bar{A})$ and any reference to $\bar{r}$ is omitted. The resulting expression is summarized as follows: (see next page)

$$
a_r(h|\overline{A},x) = \begin{cases}
a(h|\overline{A}) & \text{for } h \leq S,\ r=1,\ \text{and } 1 < x+1 \leq N; \\[2ex]
H_5 + \displaystyle\sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} P\{D(r'-2)=\nu\} \Bigg[\sum_{i=h+S-RP}^{S} a(i|\overline{A}) P\{X > r-r'\} G_1(i,r',\nu) & \\
\qquad + \displaystyle\sum_{i=h}^{S} a(i|\overline{A}) P\{X \leq r-r'\} G_2(i,r',\nu) \Bigg] & \\
 & \text{for } h \leq RP,\ 2 \leq r \leq x+1 \leq N,\ \text{and } 2 \leq \Lambda \leq N; \\[2ex]
\displaystyle\sum_{i=h}^{S} a(i|\overline{A}) P\{D(r-1)=i-h\} & \\
\qquad + \displaystyle\sum_{i=h}^{S} \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} a(i|\overline{A}) P\{D(r'-2)=\nu\} P\{X \leq r-r'\} G_2(i,r',\nu) & \\
 & \text{for } RP < h \leq S,\ 2 \leq r \leq x+1 \leq N,\ \text{and } 2 \leq \Lambda \leq N;
\end{cases} \tag{63}
$$

$$\text{where } G_1(i,r',\nu) = \sum_{\delta=0}^{(i-h)-(S-RP)} P\{D(1)=S-RP-\nu+\delta\}P\{D(r-r')=i-(S-RP+\delta)-h\}, \tag{63a}$$

$$G_2(i,r',\nu) = P\{D(1) \geq S-RP-\nu\}P\{D(r-r')=i-h\}, \tag{63b}$$

$$\text{and } H_5 = \begin{cases} \displaystyle\sum_{i=h}^{S} \sum_{\nu=0}^{S-RP-1} a(i|\bar{A})P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda-1)=i-\nu-h\} & \text{for } 2 \leq \Lambda < r \leq N, \\ \displaystyle\sum_{i=h}^{h+(S-RP-1)} a(i|\bar{A})P\{D(r-1)=i-h\} & \text{for } 2 \leq r \leq \Lambda \leq N. \end{cases} \tag{63c}$$

For Case II with $r \geq x+1$, the beginning cycle replenishment order has been received. Consequently, there are S items from which demand during the first r periods is made. This development is identical with the development of $a_r(h|A\bar{B},x,\bar{r})$ for $r \geq \bar{r}$ and $r \geq x+1$. Therefore, $a_r(h|\bar{A},x)$ for $r \geq x+1$ is the same expression as $a_r(h|A\bar{B},x,\bar{r})$ for $r \geq x+1$ and $r \geq \bar{r}$. See expression (62).

Development of the period stock level conditional probabilities, $a_r(h|AB,x)$.--The conditional probabilities that the stock level is equal to h at the beginning of period r, $a_r(h|AB,x)$, given that a within-the-cycle replenishment order was placed and received during the previous cycle and given that the replenishment lead time for the beginning cycle order is x, are denoted by $a_r(h|AB,x)$. These probabilities will be determined in terms of the beginning stock level conditional stationary probabilities, $a(i|AB)$. Consider the cases corresponding to the two mutually exclusive and exhaustive sets: $r < x+1$ and $r \geq x+1$.

For Case I with $r < x+1$, the beginning cycle replenishment order has not been received. Also since the set AB does not result in a replenishment order outstanding, any reference to the period in which a within-the-cycle replenishment order is received during the subsequent cycle is not relevant in the development of $a_r(h|AB,x)$. Consequently, this development is similar to the development of $a_r(h|A\bar{B},x,\bar{r})$ which was summarized in expression (59) for $r < \bar{r}$ and $r < x+1$ except that the expression $a(i|A\bar{B},\bar{r})$ is replaced by $a(i|AB)$ and any reference to $\bar{r}$ is omitted. The resulting expression is summarized as follows: (see next page)

$$
a_r(h|AB,x) = \begin{cases}
a(h|AB) \qquad \text{for } h \leq S,\ r=1, \text{ and } 1 < x+1 \leq N; & (64) \\[2ex]
H_6 + \displaystyle\sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} P\{D(r'-2)=\nu\} \Bigg[\sum_{i=h+S-RP}^{S} a(i|AB)P\{X > r-r'\}G_1(i,r,\nu) & \\
\qquad + \displaystyle\sum_{i=h}^{S} a(i|AB)P\{X \leq r-r'\}G_2(i,r',\nu) \Bigg] & \\
\qquad\qquad \text{for } h \leq RP,\ 2 \leq r \leq x+1 \leq N, \text{ and } 2 \leq \Lambda \leq N; & \\[2ex]
\displaystyle\sum_{i=h}^{S} a(i|AB)P\{D(r-1)=i-h\} & \\
\qquad + \displaystyle\sum_{i=h}^{S} \sum_{r'=2}^{\min(\Lambda,r)} \sum_{\nu=0}^{S-RP-1} a(i|AB)P\{D(r'-2)=\nu\}P\{X \leq r-r'\}G_2(i,r',\nu) & \\
\qquad\qquad \text{for } RP < h \leq S,\ 2 \leq r \leq x+1 \leq N, \text{ and } 2 \leq \Lambda \leq N; &
\end{cases}
$$

$$\text{where } G_1(i,r',\nu) = \sum_{\delta=0}^{(i-h)-(S-RP)} P\{D(1)=S-RP-\nu+\delta\}P\{D(r-r')=i-(S+RP+\delta)-h\}, \tag{64a}$$

$$G_2(i,r',\nu) = P\{D(1) \geq S-RP-\nu\}P\{D(r-r')=i-h\}, \tag{64b}$$

$$\text{and } H_6 = \begin{cases} \displaystyle\sum_{i=h}^{S} \sum_{\nu=0}^{S-RP-1} a(i|AB)P\{D(\Lambda-1)=\nu\}P\{D(r-\Lambda-1)=i-\nu-h\} & \text{for } 2 \leq \Lambda < r \leq N, \\ \displaystyle\sum_{i=h}^{h+(S-RP-1)} a(i|AB)P\{D(r-1)=i-h\} & \text{for } 2 \leq r \leq \Lambda \leq N. \end{cases} \tag{64c}$$

For Case II with $r < x+1$, the beginning cycle replenishment order has been received. Consequently, there are S items from which the demand during the first r periods is made. This development is identical with the development of $a_r(h|A\bar{B},x,\bar{r})$ for $r \geq \bar{r}$ and $r \geq x+1$. Therefore, $a_r(h|AB,x)$ for $r \geq x+1$ is the same expression as $a_r(h|A\bar{B},x,\bar{r})$ for $r \geq x+1$ and $r \geq \bar{r}$. See expression (62).

<u>Period stock level unconditional stationary probabilities</u>.--The unconditional probabilities for the period stock level can be obtained by inserting expressions (59), (60), (61) and (62) into expression (31); by inserting expressions (62) and (63) into expression (29); by inserting expressions (62) and (64) into expression (30); and by inserting expressions (29), (30), (31), (12), (18), and (23) into expression (28).

Results

The expressions for the stock level probabilities for the combination inventory policy have been developed under the hypothesis of back-orders allowed. The results of this chapter will be used in Chapter VII in the determination of the measures of effectiveness required to attain the primary objective of this study.

CHAPTER VII

MEASURES OF EFFECTIVENESS

Introduction

The objective of this chapter is to determine mathematical expressions for useful measures of effectiveness associated with the fixed cycle inventory policy, the (s,S) inventory policy, the variable cycle inventory policy, and the combination inventory policy. This determination will utilize the stock level probabilities developed in Chapters III, IV, V, and VI. The measures of effectiveness treated in this chapter are:

1. probability of one or more shortages,
2. expected number of shortages,
3. expected intensity of shoratages,
4. expected inventory, and
5. expected number of replenishment orders.

These particular measures of effectiveness were selected for study, rather than other measures of effectiveness, because of their seeming usefulness in facilitating inventory decisions. The literature search revealed, both explicitly and implicitly, the importance attached to these measures by theorists and practitioners. Further evidence of the usefulness of these measures will be presented in Chapter VIII in which they will become the basis for the development of decision procedures.

The probability of one or more shortages will be preferred to a similar and sometimes used measure of effectiveness, the probability of a stock-out. A stock-out implies zero stock available; unsatisfied demand does not necessarily exist. A shortage implies demand in excess of available stock. The probability of one or more shortages during the order cycle will be preferred in this study to the probability of a stock-out because the absence of stock causes inconvenience only if demand occurs simultaneously.

The expected number of shortages during the order cycle is a measure of unsatisfied demand.

The expected intensity of shortages is a combination measure of the number of shortages and the duration of shortages during the order cycle. A shortage for a limited time duration may cause less disturbance than a shortage which exists over an extended period of time.

The expected inventory is an indicator of the average magnitude of the items maintained in inventory throughout the order cycle.

The expected number of replenishment orders placed during an order cycle is indicative of the frequency with which replenishment orders are placed. This measure will be determined separately for routine replenishment orders and for special (non-routine) replenishment orders.

Throughout the analytical development of these measures of effectiveness, no distinction will be made in notation for the stock level probabilities for the two hypothesis concerning back-orders nor for the four policies treated, unless analytically necessary. It will be understood that the stock level probabilities are in correspondence with the appropriate back-order hypothesis and with the applicable inventory policy.

Probability of One or More Shortages

The objective of this section is to determine the probability of one or more shortages during the order cycle, P{L}, for the fixed cycle inventory policy, the (s,S) inventory policy, and the variable cycle inventory policy.

The determination of the probability of one or more shortages for the combination inventory policy involves analytical complexity beyond the scope of this study. However, the average probability of one or more shortages during the periods in the order cycle will be developed specifically for the combination inventory policy. Such a measure will be used in the study of decision procedures.

Inventory decisions are sometimes made subject to the constraint that the probability of one or more shortages must be some specified value. Therefore, knowledge of the probability of one or more shortages is useful in making inventory decisions.

Fixed Cycle Inventory Policy and the (s,S) Inventory Policy

The probability of one or more shortages during an order cycle will be determined first for the more difficult policy, the (s,S) inventory policy. From this expression and by making specific observations, the expression for the probability of one or more shortages for the fixed cycle inventory policy will be determined by inspection.

The (s,S) inventory policy.--The probability of one or more shortages during the order cycle is denoted by P{L}. Consider the sets corresponding to (1) a shortage of one or more items occurring and a replenishment order not being placed and (2) a shortage of one or more items occurring and a replenishment order being placed. Denote these sets by $L, \bar{\chi}$ and L, χ, respectively. These two sets are mutually exclusive and exhaustive. Therefore,

$$P\{L\} = P\{L, \bar{\chi}\} + P\{L, \chi\}. \tag{1}$$

Figure 18 (page 173) illustrates the paths in which a shortage may occur during the order cycle.

I. Case I: No replenishment order placed.--The probability of one or more shortages and that a replenishment order is not placed is denoted by $P\{L, \bar{\chi}\}$. When the beginning stock level is greater than s, there is no replenishment order placed. A shortage occurs when no replenishment order is placed, if the beginning stock level is equal to i, for $s < i \leq S$, and if the demand during the entire cycle is greater than i. The probability of this specific path is

$$a(i)P\{D(N) > i\} \quad \text{for } s < i \leq S. \tag{2}$$

Since the beginning stock level can be any of the possible values from s+1 to S inclusive,

$$P\{L, \bar{\chi}\} = \sum_{i=s+1}^{S} a(i)P\{D(N) > i\}. \tag{3}$$

II. Case II: Replenishment order placed.--The probability of one or more shortages and that a replenishment order is placed is denoted by $P\{L, \chi\}$. These probabilities will be determined from N conditional probabilities of a shortage, given a specific replenishment lead

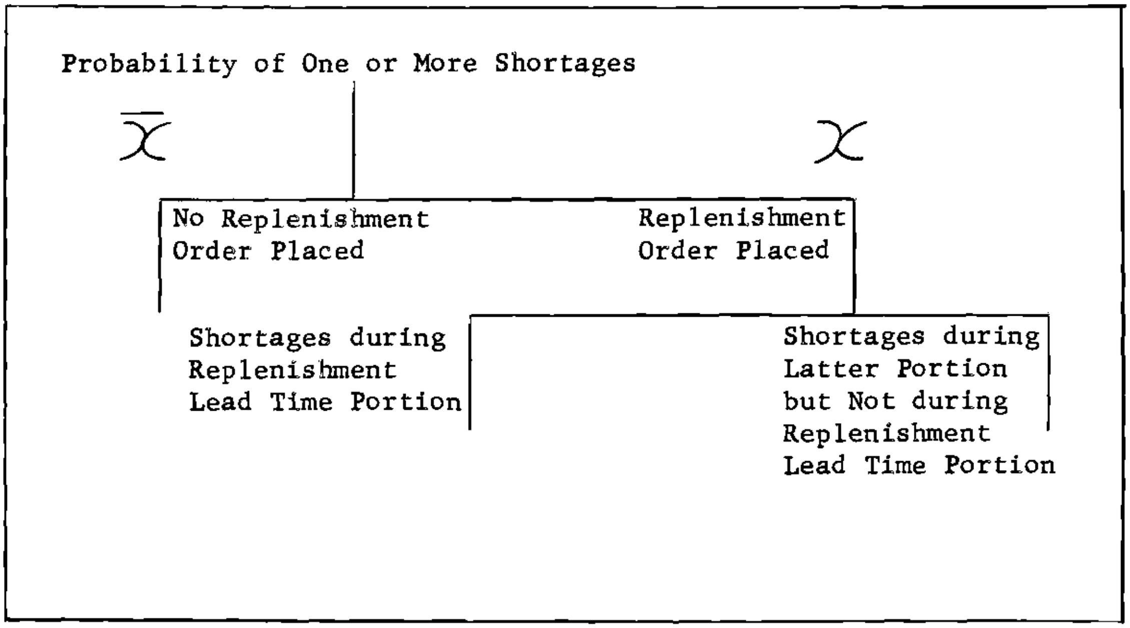

Fig. 18. Tree Diagram Illustrating the Mutually Exclusive and Exhaustive Paths in which a Shortage May Occur during the Order Cycle for the (s,S) Inventory Policy

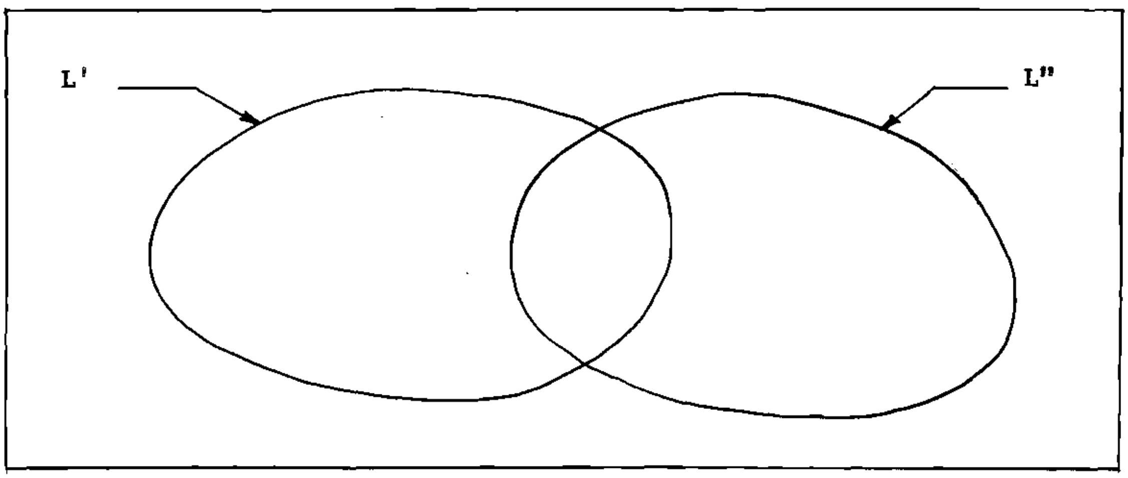

Fig. 19. Venn Diagram Illustrating the Sets of All Possible Paths in which Shortages May Occur during the Order Cycle for a Positive Lead Time

time, $P\{L|x\}$, in conjunction with the probability of this specific lead time. These conditions are mutually exclusive and exhaustive. That is,

$$P\{L, \mathcal{X}\} = \sum_{x=0}^{N-1} P\{L|x\}P\{X=x\}. \qquad (4)$$

The conditional probability, $P\{L|x\}^*$, will be determined by considering the order cycle as dichotomized into the replenishment lead time portion and into the portion after the replenishment order is received. Let $L'|x$ and $L''|x$ denote the sets of all possible paths in which one or more shortages can occur during the replenishment lead time portion, x, and in the portion after the replenishment order is received, N-x, respectively (see Figure 19 on page 173). The following development considers the general possibility that the intersection of $L'|x$ and $L''|x$ is not necessarily empty. Possible expressions which could be used for developing the conditional probability of one or more shortages during the order cycle are

$$P\{L|x\} = P\{L'|x\} + P\{L''-L'|x\} \quad \text{for } 0 \leq x \leq N-1, \qquad (5)$$

$$P\{L|x\} = P\{L'|x\} + P\{L''|x\} - P\{L'L''|x\} \quad \text{for } 0 \leq x \leq N-1, \qquad (5a)$$

$$P\{L|x\} = P\{L'-L''|x\} + P\{L''|x\} \quad \text{for } 0 \leq x \leq N-1, \qquad (5b)$$

$$P\{L|x\} = P\{L'-L''|x\} + P\{L'L''|x\} + P\{L''-L'|x\} \qquad (5c)$$
$$\text{for } 0 \leq x \leq N-1.$$

---

*For simplicity of notation in the expression $P\{L|x\}$, the symbol x indicates that the replenishment lead time is equal to x.

Expression (5) is preferred for computing the probability of one or more shortages during the cycle because of its relative simplicity. The probabilities of the two mutually exclusive and exhaustive terms in expression (5) are developed separately.

A. Development of $P\{L'|x\}$.--The probability of one or more shortages during the replenishment lead time portion of the order cycle, given a replenishment lead time of x periods, is denoted by $P\{L'|x\}$. This probability will be determined by considering two general paths corresponding to the mutually exclusive and exhaustive sets: $1 \leq i \leq s$ and $i \leq 0$.

1. Path I: $1 \leq i \leq s$.--A specific path in which a shortage of one or more items during the replenishment lead time portion occurs if the beginning stock level is equal to i, for $1 \leq i \leq s$, and if the demand during the replenishment lead time is greater than i. The probability of one or more shortages by this specific path is

$$a(i)P\{D(x) > i\} \qquad \text{for } 1 \leq i \leq s \text{ and } 0 \leq x \leq N-1. \tag{6}$$

The left side of Figure 20 (page 176) illustrates this general path.

The beginning stock level can be any of the possible values from 1 to s inclusive. Therefore, the probability of a shortage for a positive beginning stock level, given a specific replenishment lead time, is

$$\sum_{i=1}^{s} a(i)P\{D(x) > i\} \qquad \text{for } 0 \leq x \leq N-1. \tag{7}$$

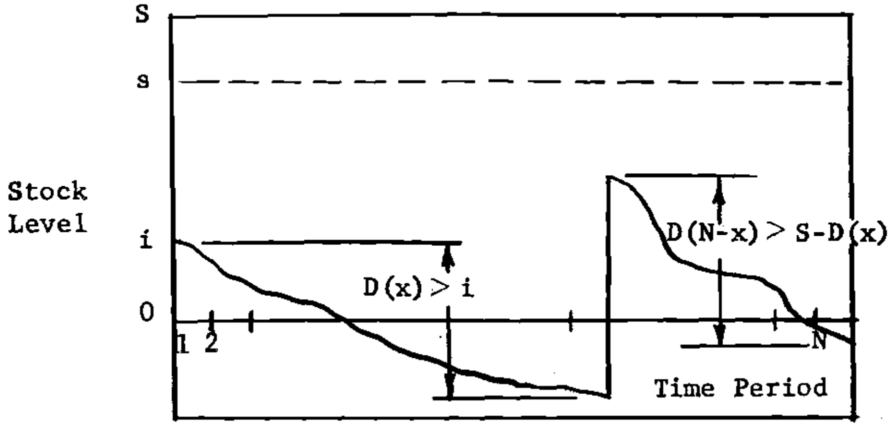

Fig. 20. General Path for Shortages for the (s,S) Inventory Policy when Replenishment Order is Placed, Back-Orders Allowed

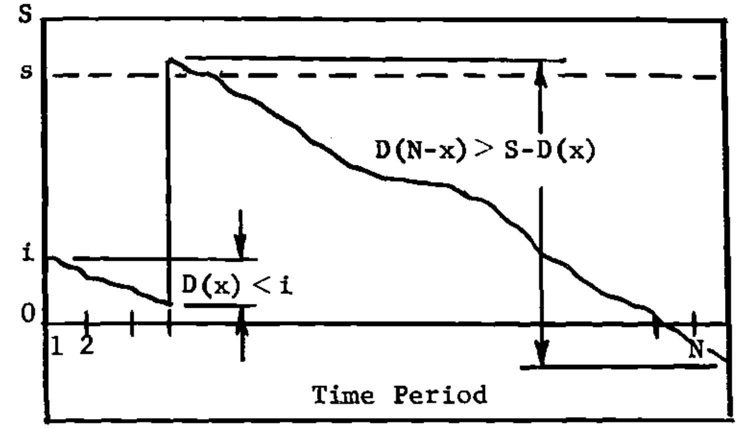

Fig. 21. General Path for Shortages during the Latter Portion of the Order Cycle but Not during the Replenishment Lead Time Portion for the (s,S) Inventory Policy when Replenishment Order is Placed, Back-Orders Allowed

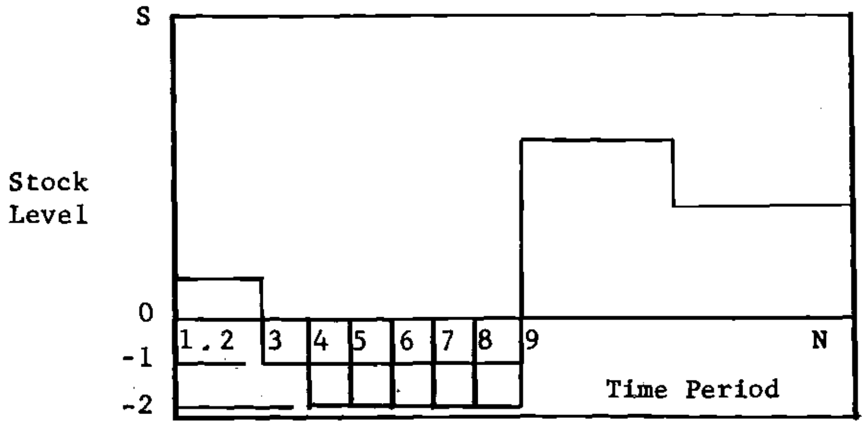

Fig. 22. Illustration of Equivalence of Intensity Expression

2. Path II: $i \leq 0$.--A shortage of one or more items during the replenishment lead time portion occurs if the beginning stock level is equal to or less than zero and if the demand during the replenishment lead time is for one or more items. The probability of one or more shortages for a non-positive beginning stock level, given a specific replenishment lead time, is

$$A(0)P\{D(x) \geq 1\} \qquad \text{for } 0 \leq x \leq N\text{-}1. \tag{8}$$

If back orders are not allowed, $A(0) = a(0)$.

3. Summary of $P\{L'|x\}$.--The probability of one or more shortages during the replenishment portion of the order cycle, given a replenishment lead time of x periods, is obtained by combining the mutually exclusive and exhaustive expressions (7) and (8) as follows:

$$P\{L'|x\} = \sum_{i=1}^{s} a(i)P\{D(x) > i\} + A(0)P\{D(x) \geq 1\} \tag{9}$$

$$\text{for } 0 \leq x \leq N\text{-}1.$$

B. Development of $P\{L''\text{-}L'|x\}$.--The probability of one or more shortages during the portion of the order cycle after the replenishment order is received but not during the replenishment lead time portion, given a replenishment lead time portion of x periods, is denoted by $P\{L''\text{-}L'|x\}$. This probability will be determined by considering two general paths corresponding to the mutually exclusive and exhaustive sets: $1 \leq i \leq s$ and $i \leq 0$.

1. Path I: $1 \leq i \leq s$.--A specific path in which one or more shortages during the portion of the order cycle after the replenishment order is received but not during the replenishment lead time portion, given a replenishment lead time of x periods, occurs if the following three independent conditions exist:

1. The beginning stock level is equal to i, for $1 \leq i \leq s$.
2. Demand during the replenishment lead time is equal to ν, for $0 \leq \nu \leq i$.
3. Demand during the latter portion of the order cycle is greater than the stock level when the replenishment order is received, $S-\nu$.

For specific values of i and ν, the probability of this path is

$$a(i)P\{D(x)=\nu\}P\{D(N-x) > S-\nu\} \qquad (10)$$

$$\text{for } 1 \leq i \leq s,\ 0 \leq \nu \leq i, \text{ and } 0 \leq x \leq N-1.$$

Figure 21 (page 176) illustrates this general path.

The beginning stock level can be any of the possible values from 1 to s inclusive. Demand during the replenishment lead time can be any of the possible values from 0 to i inclusive. Therefore, the probability of Path I is

$$\sum_{i=1}^{s} \sum_{\nu=0}^{i} a(i)P\{D(x)=\nu\}P\{D(N-x) > S-\nu\} \qquad (11)$$

$$\text{for } 0 \leq x \leq N-1.$$

2. Path II: $i \leq 0$.--A specific path in which one or more shortages during the portion of the order cycle after the replenishment

order is received but not during the replenishment lead time portion, given a replenishment lead time of x periods, occurs if the following three independent conditions exist:

1. The beginning stock level is equal to i, for $i \leq 0$.
2. Demand during the replenishment lead time is equal to zero.
3. Demand during the latter portion of the order cycle is greater than the order level, S.

Therefore, the probability of Path II is

$$A(0)P\{D(x)=0\}P\{D(N-x) > S\} \qquad (12)$$

$$\text{for } 0 \leq x \leq N-1.$$

3. Expression for $P\{L''-L' | x\}$.--The probability of one or more shortages during the portion of the order cycle after the replenishment order is received but not during the replenishment lead time portion, given a replenishment lead time of x periods, is obtained by combining expressions (11) and (12) as follows:

$$P\{L''-L' | x\} = \sum_{i=1}^{S} \sum_{\nu=0}^{i} a(i)P\{D(x)=\nu\}P\{D(N-x) > S-\nu\} \qquad (13)$$

$$+ A(0)P\{D(x)=0\}P\{D(N-x) > S\} \qquad \text{for } 0 \leq x \leq N-1.$$

III. Probability of one or more shortages during the order cycle.--The probability of one or more shortages during the order cycle is obtained by inserting expressions (3) and (4) into expression (1). Expression (4) is obtained from expression (5), which is obtained from expressions (9) and (13). Therefore,

$$P\{L\} = \sum_{i=s+1}^{S} a(i)P\{D(N) > i\} + \sum_{x=0}^{N-1} \Bigg[\sum_{i=1}^{s} a(i)P\{D(x) > i\} \qquad (14)$$

$$+ \; A(0)P\{D(x) \geq 1\} + \sum_{i=1}^{s} \sum_{\nu=0}^{i} a(i)P\{D(x)=\nu\}P\{D(N-x) > S-\nu\}$$

$$+ \; A(0)P\{D(x)=0\}P\{D(N-x) > S\} \Bigg] P\{X=x\} \quad \text{for } 0 \leq x \leq N-1.$$

<u>Fixed cycle inventory policy</u>.--In the development of the fixed cycle inventory policy, a replenishment order is placed at the beginning of each order cycle. A replenishment order for zero items was considered meaningful in the analytical development of the stock level probabilities. The (s,S) inventory policy is the fixed cycle inventory policy when s=S. Therefore, if S is substituted for s in expression (14), the expression for the probability of one or more shortages during the order cycle for the fixed cycle inventory policy is as follows:

$$P\{L\} = \sum_{x=0}^{N-1} \Bigg[\sum_{i=1}^{S} a(i)P\{D(x) > i\} + A(0)P\{D(x) \geq 1\} \qquad (15)$$

$$+ \sum_{i=1}^{S} \sum_{\nu=0}^{i} a(i)P\{D(x)=\nu\}P\{D(N-x) > S-\nu\}$$

$$+ \; A(0)P\{D(x)=0\}P\{D(N-x) > S\} \Bigg] P\{X=x\} \quad \text{for } 0 \leq x \leq N-1.$$

Variable Cycle Inventory Policy

The probability of one or more shortages during an order cycle, P{L}, will be determined for the variable cycle inventory policy. This probability will be determined from N conditional probabilities of a shortage, given a specific replenishment lead time, in conjunction with the probability of this specific lead time. These conditions are mutually exclusive and exhaustive. That is,

$$P\{L\} = \sum_{x=0}^{K} P\{L|x\}P\{X=x\}. \tag{16}$$

As was the case with the (s,S) inventory policy, consider the variable order cycle as dichotomized into the replenishment lead time portion and into the portion after the replenishment order is received. Let L' and L'' denote the sets of all possible paths in which one or more shortages can occur during the replenishment lead time portion and in the portion after the replenishment order is received, respectively (see Figure 19 on page 176).

The conditional probability of one or more shortages during the order cycle, given a specific replenishment lead time, will be determined from the expression

$$P\{L|x\} = P\{L'|x\} + P\{L''-L'|x\} \tag{17}$$
$$\text{for } 0 \leq x \leq K.$$

As a consequence of (1) the last period in the variable cycle being greater than K periods and (2) the variable cycle inventory policy ordering rule, shortages in the set L'' occur only during the last period, ω, for $\omega > K$, of the variable length cycle. Since the replenishment lead time is equal to or less than K periods, any shortages in the set L' occur during the first K periods.

Development of $P\{L'|x\}$.--The probability of one or more shortages during the replenishment lead time portion of the order cycle is denoted by

$P\{L'|x\}$. This probability can be determined by considering the two general paths corresponding to the mutually exclusive and exhaustive sets: $1 \leq i \leq RP$ and $i \leq 0$. The development of $P\{L'|x\}$ for the variable cycle inventory policy is similar to the development of $P\{L'|x\}$ for the (s,S) inventory policy except that (1) s is replaced by RP and that (2) the upper range of x is K rather than N-1. The resulting expression is summarized as follows:

$$P\{L'|x\} = \sum_{i=1}^{RP} a(i)P\{D(x) > i\} + A(0)P\{D(x) > i\} \tag{18}$$

$$\text{for } 0 \leq x \leq K.$$

<u>Development of $P\{L''-L'|x\}$</u>.--The probability of one or more shortages after the replenishment order is received but not during the replenishment lead time portion, given a replenishment lead time of x periods, is denoted by $P\{L''-L'|x\}$. This probability will be developed from the following expression:

$$P\{L''-L'|x\} = \sum_{\omega=K+1}^{\infty} P\{L''-L', \Omega=\omega|x\} \tag{19}$$

$$\text{for } 0 \leq x \leq K.$$

In the development of $P\{L''-L', \Omega=\omega|x\}$, consider the two general paths corresponding to the mutually exclusive and exhaustive sets: $1 \leq i \leq RP$ and $i \leq 0$.

I. <u>Path I: $1 \leq i \leq RP$</u>.--A specific path in which there are one or more shortages in the set $L''-L'$ and in which the length of cycle is

ω periods, given a replenishment lead time of x periods, occurs if the following four independent conditions exist:

1. The beginning stock level is equal to i, for $1 \leq i \leq RP$.
2. Demand during the replenishment lead time is equal to ν, for $0 \leq \nu \leq i$.
3. Demand during the next ω-x-1 periods is equal to ξ, for $0 \leq \xi \leq S\text{-}RP\text{-}1\text{-}\nu$.
4. Demand during period ω is greater than $S\text{-}\nu\text{-}\xi$.

For specific values of i, ν, and ξ, the probability of this path is

$$a(i)P\{D(x)=\nu\}P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\nu-\xi\} \tag{20}$$

$$\text{for } 1 \leq i \leq RP,\ 0 \leq \nu \leq i,\ 0 \leq \xi \leq S-RP-1-\nu, \text{ and } 0 \leq x \leq K.$$

Figure 21 (page 176) illustrates a similar path for the (s,S) inventory policy.

The beginning stock level can be any of the possible values from 1 to RP inclusive. Demand during the replenishment lead time can be any of the possible values from 0 to i inclusive. Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1-ν inclusive. Therefore, the probability of Path I is

$$\sum_{i=1}^{RP} \sum_{\nu=0}^{i} \sum_{\xi=0}^{S-RP-1-\nu} a(i)G(\nu,\xi), \tag{21}$$

$$\text{where } G(\nu,\xi) = P\{D(x)=\nu\}P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\nu-\xi\}$$

$$\text{for } 0 \leq x \leq K.$$

II. <u>Path II: $i \leq 0$</u>.--A specific path in which there are one or more shortages in the set L"-L' and in which the length of cycle is ω periods, given a replenishment lead time of x periods, occurs if the following four independent conditions exist:

1. The beginning stock level is equal to i, for $i \leq 0$.
2. Demand during the replenishment lead time is equal to zero.
3. Demand during the next ω-x-1 periods is equal to ξ, for $0 \leq \xi \leq S-RP-1$.
4. Demand during period ω is greater than S-ξ.

For a specific value of ξ, the probability of Path II is

$$A(0)P\{D(x)=0\}P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\xi\} \qquad (22)$$

$$\text{for } 0 \leq \xi \leq S\text{-}RP\text{-}1 \text{ and } 0 \leq x \leq K.$$

Demand during the ω-x-1 periods can be any of the possible values from 0 to S-RP-1 inclusive. Therefore, the probability of Path II is

$$A(0)P\{D(x)=0\} \sum_{\xi=0}^{S-RP-1} P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\xi\} \qquad (23)$$

$$\text{for } 0 \leq x \leq K.$$

III. <u>Summary of P{L"-L'|x}</u>.--The probability of one or more shortages after the replenishment order is received but not during the replenishment lead time portion, given a replenishment lead time of x periods, is obtained by combining expressions (21) and (23) and inserting in expression (19) as follows:

$$P\{L''-L'|x\} = \sum_{\omega=K+1}^{\infty}\left[\sum_{i=1}^{RP}\sum_{\nu=0}^{i}\sum_{\xi=0}^{S-RP-1-\nu} a(i)G(\nu,\xi)\right. \tag{24}$$

$$\left. + A(0)P\{D(x)=0\}\sum_{\xi=0}^{S-RP-1} P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\nu\}\right], \tag{24a}$$

where $G(\nu,\xi)=P\{D(x)=\nu\}P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\nu-\xi\}$,

for $0 \leq x \leq K$.

Probability of one or more shortages during the variable order cycle.-- The probability of one or more shortages during the variable order cycle is obtained by inserting expressions (18) and (24) into expression (17) which is inserted into expression (16) as follows:

$$P\{L\} = \sum_{x=0}^{K}\left[\sum_{i=1}^{RP} a(i)P\{D(x) > i\} + A(0)P\{D(x) \geq 1\}\right. \tag{25}$$

$$+ \sum_{\omega=K+1}^{\infty}\sum_{i=1}^{RP}\sum_{\nu=0}^{i}\sum_{\xi=0}^{S-RP-1-\nu} a(i)G(\nu,\xi)$$

$$\left. + A(0)P\{D(x)=0\}\sum_{\omega=K+1}^{\infty}\sum_{\xi=0}^{S-RP-1} P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\nu\}\right] P\{X=x\},$$

where $G(\nu,\xi)=P\{D(x)=\nu\}P\{D(\omega-x-1)=\xi\}P\{D(1) > S-\nu-\xi\}$. (25a)

Combination Inventory Policy

As indicated previously, the determination of the probability of one or more shortages for the combination inventory policy involves analytical complexity beyond the scope of the present study. However, the average probability, $P\{\bar{L}\}$, of one or more shortages during the periods in the order cycle may be useful. This average probability will be obtained from the expression

$$P\{\bar{L}\} = \frac{1}{N} \sum_{r=1}^{N} P\{L_r\} \quad . \tag{26}$$

The probability of one or more shortages during any period, $P\{L_r\}$, will be determined by considering the two mutually exclusive and exhaustive general paths: $1 \leq h \leq S$ and $h \leq 0$.

<u>Path I: $1 \leq h \leq S$</u>.--A specific path in which a shortage of one or more items during any period r for a positive stock level occurs if the demand during this one period, $D(1)$, is greater than the beginning stock level, h. The probability of this specific path is

$$a_r(h)P\{D(1) > h\} \qquad \text{for } 1 \leq h \leq S. \tag{27}$$

The stock level can be any of the possible values from 1 to S inclusive. Therefore, the probability of Path I is

$$\sum_{h=1}^{S} a_r(h)P\{D(1) > h\}. \tag{28}$$

<u>Path II: $h \leq 0$</u>.--A specific path in which a shortage of one or more items during any period r for a non-positive stock level occurs if the demand during the period is for one or more items. The probability of Path II is

$$A_r(0)P\{D(1) \geq 1\}. \tag{29}$$

Summary of $P\{L_r\}$.--The probability of one or more shortages during any period is obtained by combining the mutually exclusive and exhaustive expressions (28) and (29) as follows:

$$P\{L_r\} = \sum_{h=1}^{S} a_r(h)P\{D(1) > h\} + A_r(0)P\{D(1) \geq 1\}. \tag{30}$$

The average probability of one or more shortages during the periods in the order cycle, $P\{\overline{L}\}$, is obtained by inserting expression (30) into expression (26) as

$$P\{\overline{L}\} = \frac{1}{N} \sum_{r=1}^{N} \left[\sum_{h=1}^{S} a_r(h)P\{D(1) > h\} + A_r(0)P\{D(1) \geq 1\} \right]. \tag{31}$$

The probability of one or more shortages for the fixed cycle inventory policy, the (s,S) inventory policy, and the variable cycle inventory policy developed in this section will be used in the study of decision procedures. For the combination inventory policy, the average probability of one or more shortages during the periods in the order cycle will be used in the study of decision procedures.

Expected Number of Shortages

The objective of this section is to determine the expected number of shortages during the order cycle, $E(L)$. When the total cost of shortages is proportional to the number of shortages, the expected number of shortages will be useful in making inventory decisions. The results of the determination of the expected number of shortages will be applicable to any of the inventory policies considered in this study.

Fixed Cycle, the (s,S), and Combination Inventory Policies

For the fixed cycle inventory policy, the (s,S) inventory policy, and the combination inventory policy, there is a constant number of periods, N, in each order cycle. Also the number of shortages L_1, L_2, . . ., L_N in periods 1, 2, . . ., N are random variables with expectations, $E(L_1)$, $E(L_2)$, . . ., $E(L_N)$. Therefore, the expectation of their sum exists and is the sum of their expectations. Hence, the expected number of shortages during the order cycle for the fixed cycle, the (s,S), and the combination inventory policies is

$$E(L) = \sum_{r=1}^{N} E(L_r). \tag{32}$$

The expected number of shortages during any period r, $E(L_r)$, will be obtained by summing the product of the number of shortages, ξ, multiplied by the probability of ξ shortages over all possible shortages. That is,

$$E(L_r) = \sum_{\xi=1}^{\infty} \xi P\{\text{shortages in period } r=\xi\} \quad \text{for } 1 \leq r \leq N, \tag{33}$$

$$\text{where } P\{\text{shortages in period } r=\xi\} = P\{D(1)=\xi\}A_r(0) \tag{33a}$$
$$+ P\{D(1)=\xi+1\}a_r(1) + \ldots + P\{D(1)=\xi+S\}a_r(S)$$
$$\text{for } 1 \leq r \leq N.$$

By writing expression (33a) more compactly,

$$P\{\text{shortages in period } r=\xi\} = P\{D(1)=\xi\}A_r(0) \tag{34}$$

$$+ \sum_{h=1}^{S} P\{D(1)=\xi+h\}a_r(h) \qquad \text{for } 1 \leq r \leq N.$$

The expected number of shortages for period r is determined by inserting expression (34) into expression (33) as follows:

$$E(L_r) = \sum_{\xi=1}^{\infty} \xi\left[P\{D(1)=\xi\}A_r(0) + \sum_{h=1}^{S} P\{D(1)=\xi+h\}a_r(h)\right] \tag{35}$$

$$\text{for } 1 \leq r \leq N.$$

Consequently, the number of shortages during the order cycle can be obtained by inserting expression (35) into expression (32).

Variable Cycle Inventory Policy

For the variable cycle inventory policy, there is a variable number of periods, Ω, in each cycle, which assumes values of $\omega=K+1$, $K+2, \ldots$.

The expected number of shortages during the variable cycle inventory policy, $E(L)$, can be obtained by weighting the expected number of shortages, given a specific length of cycle, $E(L|\omega)$, by the probability of the specific length of cycle. That is,

$$E(L) = \sum_{\omega=K+1}^{\infty} E(L|\omega)P\{\Omega=\omega\}. \tag{36}$$

The number of shortages $L_1|\omega$, $L_2|\omega$, . . ., $L_\omega|\omega$ in periods 1, 2, . . ., ω are random variables with expectations $E(L_1|\omega)$, $E(L_2|\omega)$, . . ., $E(L_\omega|\omega)$. Therefore, for a specific value of ω, the expectations of their sum exists and is the sum of their expectations. For a specific length of cycle, the expected number of shortages during the variable cycle is

$$E(L|\omega) = \sum_{r=1}^{\omega} E(L_r|\omega) \qquad \text{for } \omega > K. \tag{37}$$

The expected number of shortages for period r, given that the length of cycle is ω, is denoted by $E(L_r|\omega)$. This expected value can be determined similarly to $E(L_r)$ in expression (35) except that the condition, length of cycle, must be specified. The resulting expression is summarized as follows:

$$E(L_r|\omega) = \sum_{\xi=1}^{\infty} \xi \Big[P\{D(1)=\xi\} A_r(0|\omega) + \sum_{h=1}^{S} P\{D(1)=\xi+h\} a_r(h|\omega) \Big] \tag{38}$$

$$\text{for } 1 \leq r \leq \omega \text{ and } \omega > K.$$

Consequently, the number of shortages during the variable order cycle can be obtained by inserting expression (38) into expression (37), which is inserted into expression (36).

The expected number of shortages developed in this section will be used in the study of decision procedures.

Expected Intensity of Shortages

The objective of this section is to determine the expected intensity of shortages during the order cycle, E(T). The expected intensity of shortages is a combination measure of the number of shortages and the duration of shortages during the order cycle. When total cost of shortages is proportional to a combination measure of the number of shortages and the duration of the shortages, the expected intensity of shortages will be useful in making inventory decisions. The results of the determination of the expected intensity of shortages will be applicable to any of the inventory policies considered in this study.

Under the hypothesis of back-orders not allowed, all demand must be fulfilled immediately. Hence, a measure which utilizes the duration of a shortage has little significance. Therefore, the expected intensity of shortages will not be used with reference to inventory situations in which back-orders are not allowed, only with reference to inventory situations in which back-orders are allowed.

The product of the number of shortages and the duration of shortages represents the area below the zero stock level axis (see Figure 22, page 176). It will be convenient to consider this area as the sum of negative stock levels at each period; that is, a horizontal summation rather than a vertical summation. The expected intensity of the shortages is the expected value of this area below the zero stock level axis. This measure will be developed for the variable cycle inventory policy separate from the inventory policies in which the number of periods in the order cycle is constant.

Fixed Cycle, the (s,S), and Combination Inventory Policies

For the inventory policies of which the number of periods in the order cycle is constant, let T_1, T_2, . . ., T_N denote the negative stock level random variables in periods 1, 2, . . ., N which have expectations $E(T_1)$, $E(T_2)$, . . ., $E(T_N)$. Therefore, the expected intensity of the shortages is the sum of the negative stock level for each of the periods in the order cycle. That is,

$$E(T) = \sum_{r=1}^{N} E(T_r). \tag{39}$$

The expected negative stock level, $E(T_r)$, for any period r is

$$E(T_r) = \sum_{h=-\infty}^{0} h[a_r(h)]. \tag{40}$$

The expected intensity of shortages for the fixed cycle inventory policy, the (s,S) inventory policy, and the combination inventory policy is obtained by inserting expression (39) into expression (40) as follows:

$$E(T) = \sum_{r=1}^{N} \sum_{h=-\infty}^{0} h[a_r(h)]. \tag{41}$$

Variable Cycle Inventory Policy

For the variable cycle inventory policy, the expected intensity of the shortages, E(T), will be developed. This measure of effectiveness will be developed by considering an infinite number of mutually exclusive and exhaustive conditional expected intensity of shortages, given a specific length of cycle, in conjunction with the probability of the length of cycle. That is,

$$E(T) = \sum_{\omega=K+1}^{\infty} E(T|\omega)P\{\Omega=\omega\}. \tag{42}$$

Let $T_1|\omega$, $T_2|\omega$, . . ., $T_\omega|\omega$ denote the negative stock level random variables in periods 1, 2, . . ., ω, given the length of the cycle, which have expectations $E(T_1|\omega)$, $E(T_2|\omega)$, . . ., $E(T_\omega|\omega)$. Therefore, the conditional expected intensity of the shortages is the sum of the expected negative stock level for each of the periods in the order cycle. For a specific length of cycle, the expected intensity of shortages during the variable cycle is

$$E(T|\omega) = \sum_{r=1}^{\omega} E(T_r|\omega), \text{ for } \omega > K. \tag{43}$$

The conditional expected negative stock level for any period r, given a specific length of cycle, is

$$E(T_r|\omega) = \sum_{h=-\infty}^{0} h[a_r(h|\omega)], \text{ for } 1 \le r \le \omega \text{ and } \omega > K. \tag{44}$$

The expected intensity of shortages for the variable cycle inventory policy is obtained by substituting expression (44) into expression (43), which is inserted into expression (42) as follows:

$$E(T) = \sum_{\omega=K+1}^{\infty} \sum_{r=1}^{\omega} \sum_{h=-\infty}^{0} h[a_r(h|\omega)]P\{\Omega=\omega\} \ . \tag{45}$$

The expected intensity of shortages developed in this section will be used in the study of decision procedures.

Expected Inventory

The objective of this section is to determine the expected inventory, $E(I)$. The expected inventory is a measure of the average magnitude of the items maintained in inventory throughout the order cycle.* The results of the determination of the expected inventory will be applicable to each of the inventory policies considered in this study. This measure will be determined for the variable cycle inventory policy separate from the inventory policies in which the number of periods in the order cycle is constant.

Fixed Cycle, the (s,S), and Combination Inventory Policies

The expected inventory throughout the order cycle, $E(I)$, is the average of the expected inventory for each of the periods within the order cycle, $E(I_r)$. That is,

$$E(I) = \frac{1}{N} \sum_{r=1}^{N} E(I_r). \tag{46}$$

The expected number of items in inventory during period r is

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*Other inventory measures of effectiveness may be useful. An inventory measure which is proportional to the product of the number of items in inventory and the duration in which the items are in inventory could be readily determined from the stock level probabilities.

$$E(I_r) = \sum_{h=1}^{S} h[a_r(h)]. \tag{47}$$

The expected inventory throughout the order cycle for the fixed cycle inventory policy, the (s,S) inventory policy, and the combination inventory policy is obtained by inserting expression (47) into expression (46) as follows:

$$E(I) = \frac{1}{N} \sum_{r=1}^{N} \sum_{h=1}^{S} h[a_r(h)]. \tag{48}$$

Variable Cycle Inventory Policy

For the variable cycle inventory policy, the expected inventory, $E(I)$, will be developed by considering an infinite number of mutually exclusive, and exhaustive conditional expected inventory measures, $E(I|\omega)$, given a specific length of cycle, in conjunction with the probability of the length of cycle. That is,

$$E(I) = \sum_{\omega=K+1}^{\infty} E(I|\omega)P\{\Omega=\omega\}. \tag{49}$$

The conditional expected inventory throughout the order cycle, $E(I|\omega)$, is the average of the conditional expected inventory for each of the periods within the order cycle, $E(I_r|\omega)$. That is,

$$E(I|\omega) = \frac{1}{\omega} \sum_{r=1}^{\omega} E(I_r|\omega) \quad \text{for } \omega > K. \tag{50}$$

The conditional expected number of items in inventory during period r, given a specific length of cycle, is

$$E(I_r|\omega) = \sum_{h=1}^{S} h[a_r(h|\omega)] \quad \text{for } 1 \leq r \leq \omega \text{ and } \omega > K. \quad (51)$$

The expected inventory for the variable cycle inventory policy is obtained by inserting expression (51) into expression (50), which is inserted into expression (49) as follows:

$$E(I) = \sum_{\omega=K+1}^{\infty} \frac{1}{\omega} \sum_{r=1}^{\omega} \sum_{h=1}^{S} h[a_r(h|\omega)]P\{\Omega=\omega\}. \quad (52)$$

The expected inventory developed in this section will be used in the study of decision procedures.

Expected Number of Replenishment Orders

The objective of this section is to determine the expected number of routine replenishment orders, E(R), and the expected number of special replenishment orders, E(R'). These measures are indicative of the frequency with which the two types of replenishment orders are placed; therefore, the expected number of replenishment orders is useful in making inventory decisions. The results of the determination of the expected number of replenishment orders in an order cycle will be applicable to each of the inventory policies considered in this study. This measure will be determined separately for each of the inventory policies considered.

Fixed Cycle Inventory Policy

A routine replenishment order is placed at the beginning of each order cycle for the fixed cycle inventory policy if the beginning stock level is less than S. That is,

$$E(R) = [0]a(S) + [1]A(S-1), \tag{53}$$

$$\text{or } E(R) = A(S-1). \tag{54}$$

The (s,S) Inventory Policy

A routine replenishment order is placed at the beginning of each order cycle for the (s,S) inventory policy if the beginning stock level is equal to or less than s. That is,

$$E(R) = [0][A(S)-A(s+1)] + [1]A(s), \tag{55}$$

$$\text{or } E(R) = A(s). \tag{56}$$

Variable Cycle Inventory Policy

A special replenishment order is placed at the beginning of the first period in which the stock level is equal to or below the reorder point. That is, an order is placed at the beginning of each variable cycle. Therefore, the expected number of replenishment orders placed during each variable order cycle is

$$E(R') = 1. \tag{57}$$

Combination Inventory Policy

Both a routine replenishment order and a special (non-routine or within-the-cycle) replenishment order may be placed during the order cycle.

A routine replenishment order is placed at the beginning of each order cycle for the combination inventory policy if the beginning stock level is less than S. That is,

$$E(R) = [0]a(S) + [1]A(S-1), \tag{58}$$

$$\text{or } E(R) = A(S-1). \tag{59}$$

A special replenishment order is placed at the beginning of the first period within the first Λ periods of the order cycle in which the sum of stock level and stock-on-order is equal to or less than the reorder point. Equivalently, a special within-the-cycle replenishment order is placed if the demand during the Λ-1 periods is equal to or greater than S-RP. That is,

$$E(R') = P\{D(\Lambda-1) \geq S-RP\}. \tag{60}$$

The expected number of replenishment orders developed in this section will be used in the study of decision procedures.

Results

The expressions for the measures of effectiveness have been determined for the fixed cycle inventory policy, the (s,S) inventory policy, the variable cycle policy, and the combination inventory policy. The determination of these expressions for the measures of effectiveness satisfy the primary objective of this study. These measures of effectiveness will be used in Chapter VIII as the basis for the development of procedures for the selection of optimal values of decision variables, for the study of the sensitivity of total relevant cost, and the choosing of the best policy from among the four selected policies.

CHAPTER VIII

DECISION PROCEDURES

Introduction

The objectives of this chapter are to develop procedures for (1) the selection of optimal values of decision procedures for a given inventory policy, (2) the study of the sensitivity of total relevant cost, and (3) the choosing of the best policy from among the four selected policies.

The development of decision procedures in this chapter assumes that the measures of effectiveness determined analytically in Chapter VII have been computed for particular cases. These computations would be completed for a wide range of values for the controlled variables, given specific demand distributions and replenishment lead time distributions. These computations would be performed in two series, according to the provision for dealing with unsatisfied demand. Figures 23 through 35 represent conceptual relationships of variables and are suggestive of the types of curves that would be useful in a practical situation. The numbers on these curves indicate the direction in which a third variable increases.

Figures 23, 24, and 25 are suggestive of the charts appropriate for the study of the fixed cycle inventory policy. In Figure 23 (page 200), $P\{L\}$, $E(L)$, and $E(T)$ would be drawn for relevant values of S and N. Figure 24 (page 200) and Figure 25 (page 200) would be drawn in a manner similar to Figure 23, for $E(I)$ and $E(R)$, respectively.

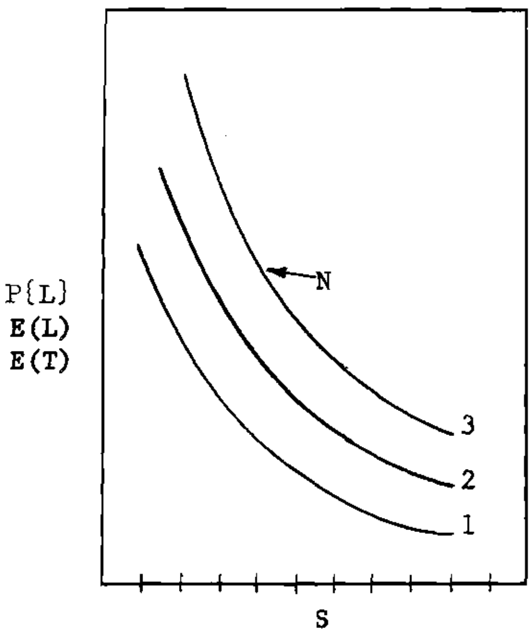

Fig. 23. Suggestive Relationship of P{L}, E(L), and E(T) to S for Values of N for the Fixed Cycle Inventory Policy

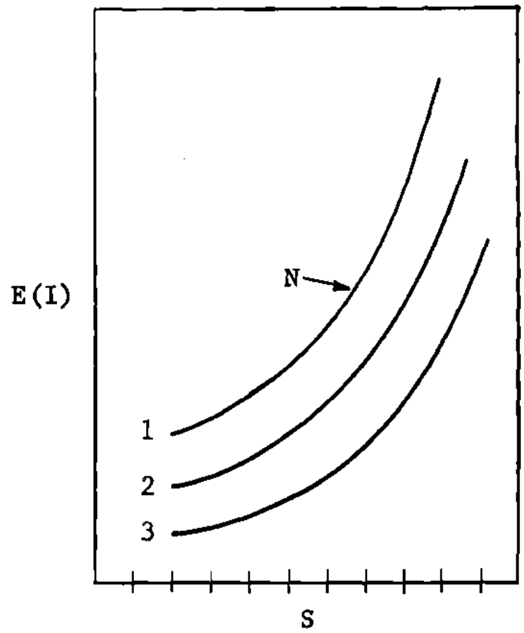

Fig. 24. Suggestive Relationship of E(I) to S for Values of N for the Fixed Cycle Inventory Policy

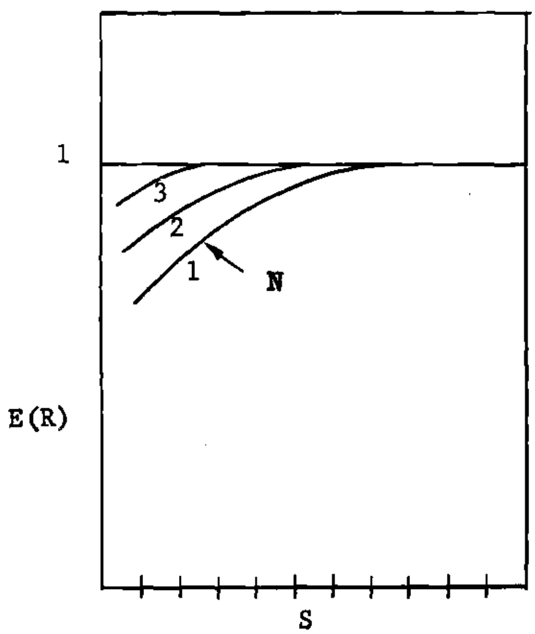

Fig. 25. Suggestive Relationship of E(R) to S for Values of N for the Fixed Cycle Inventory Policy

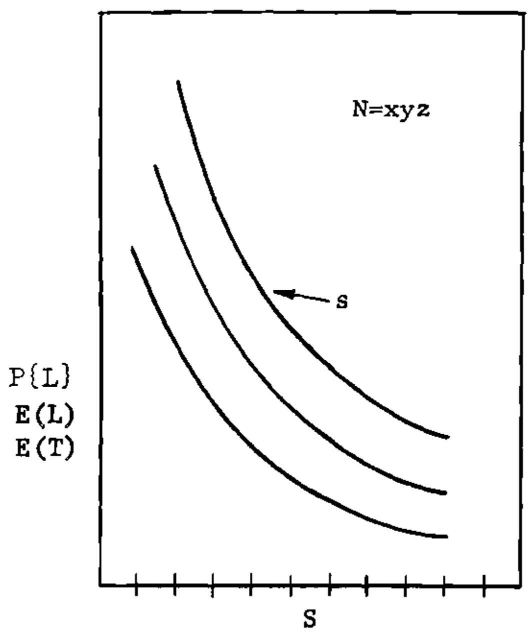

Fig. 26. Suggestive Relationship of P{L}, E(L), and E(T) to S for Values of s with N specified for the (s,S) Inventory Policy

Figures 26, 27, and 28 are suggestive of the charts appropriate for the study of the (s,S) inventory policy. In Figure 26 (page 200), P{L}, E(L), and E(T) would be drawn for relevant values of s and S. Since Figure 26 (page 200) would be drawn for a specific value of N, additional charts similar to Figure 26 would be necessary for obtaining these measures of effectiveness corresponding to other relevant values of N. Figure 27 (page 202) and Figure 28 (page 202) would be drawn in a manner similar to Figure 26 for E(I) and E(R), respectively.

Figures 29, 30, 31, and 32 are suggestive of the charts appropriate for the study of the variable cycle inventory policy. In Figure 29 (page 202), P{L}, E(L), and E(T) would be drawn for relevant values of RP and S. Figure 30 (page 202), Figure 31 (page 203), and Figure 32 (page 203) would be drawn in a manner similar to Figure 29 for E(I), E(R), and E(Ω), respectively.

Figures 33, 34, 35, and 36 are suggestive of the charts appropriate for the study of the combination inventory policy. In Figure 33 (page 203), P{L}, E(L), and E(T) would be drawn for specific values of RP and S. Since Figure 33 is for a specific value of N and Λ, additional charts similar to Figure 33 would be necessary for obtaining these measures of effectiveness corresponding to other relevant values of N and Λ. Figure 34 (page 203) and Figure 35 (page 204) would be drawn in a manner similar to Figure 33 for E(I) and E(R), respectively. In Figure 36 (page 204), E(R') would be drawn for relevant values of S-RP and Λ.

Throughout the development of the decision procedures in this chapter, the following special notation will be used:

p_i A specific value of the i^{th} decision variable.

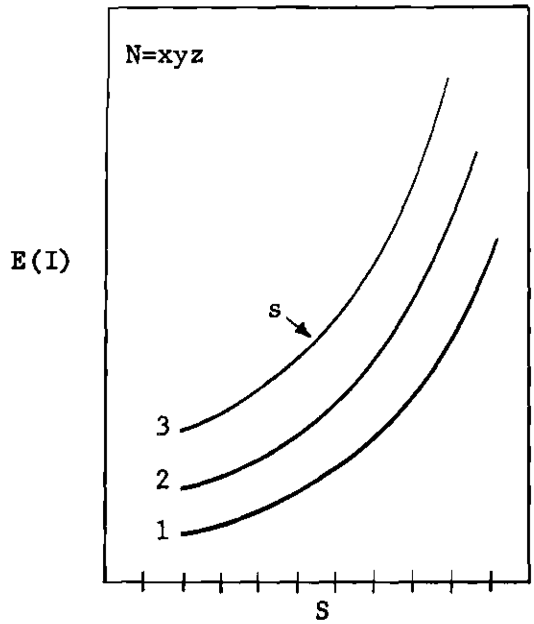

Fig. 27. Suggestive Relationship of E(I) to S for Values of s with N Specified for the (s,S) Inventory Policy

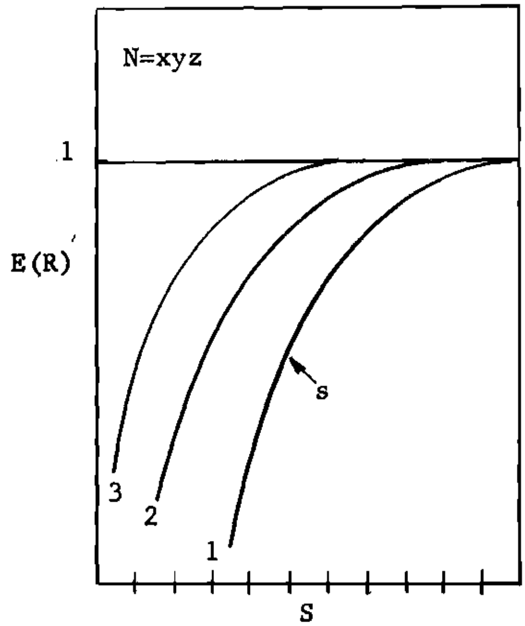

Fig. 28. Suggestive Relationship of E(R) to S for Values of s with N Specified for the (s,S) Inventory Policy

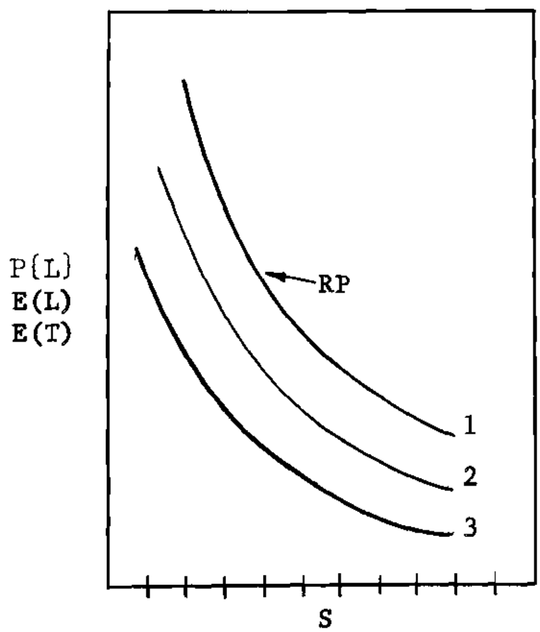

Fig. 29. Suggestive Relationship of P{L}, E(L), and E(T) to S for Values of RP for the Variable Cycle Inventory Policy

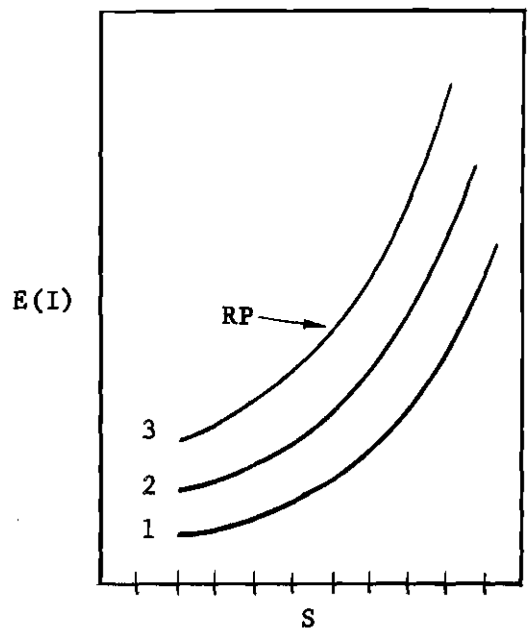

Fig. 30. Suggestive Relationship of E(I) to S for Values of RP for the Variable Cycle Inventory Policy

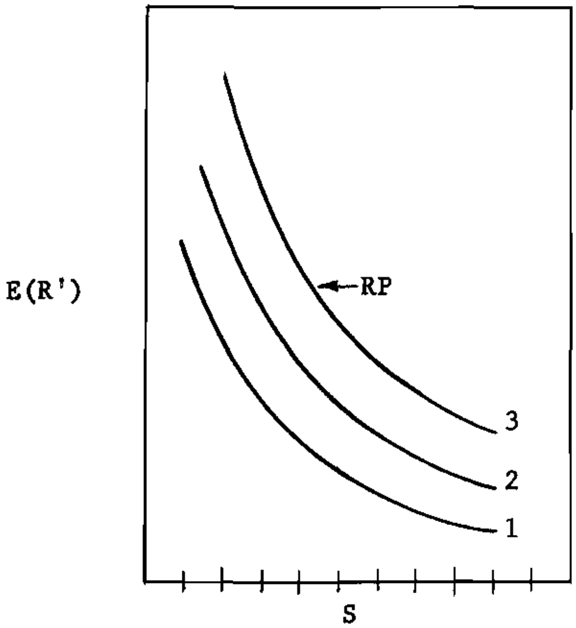

Fig. 31. Suggestive Relationship of E(R') to S for Values of RP for the Variable Cycle Inventory Policy

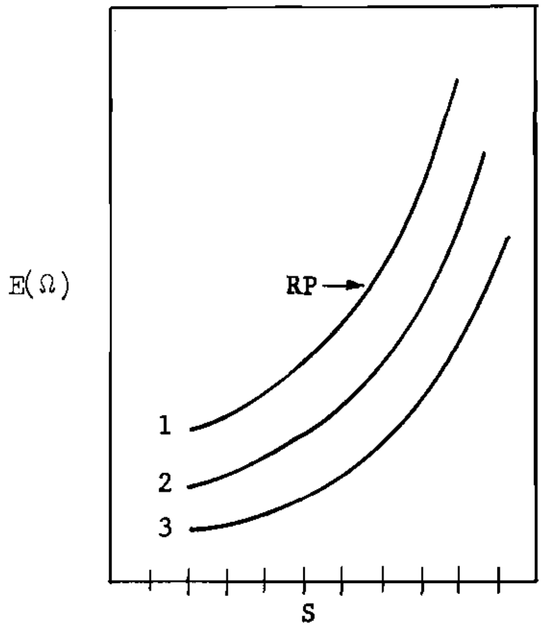

Fig. 32. Suggestive Relationship of E(Ω) to S for Values of RP for the Variable Cycle Inventory Policy

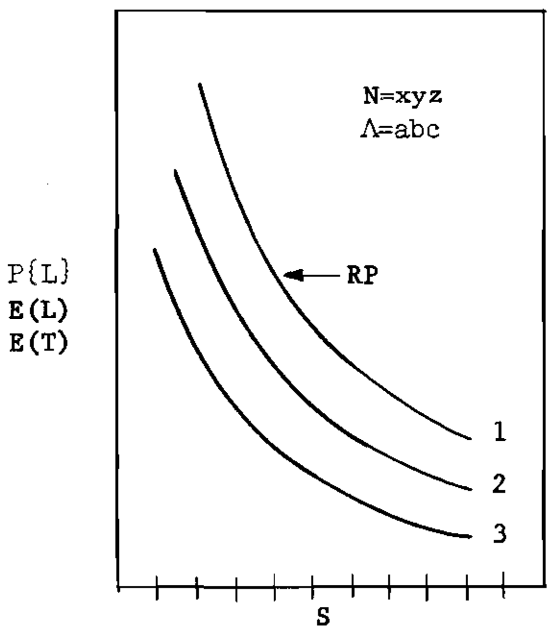

Fig. 33. Suggestive Relationship of P{L}, E(L), and E(T) to S for Values of RP with N and Λ Specified for the Combination Inventory Policy

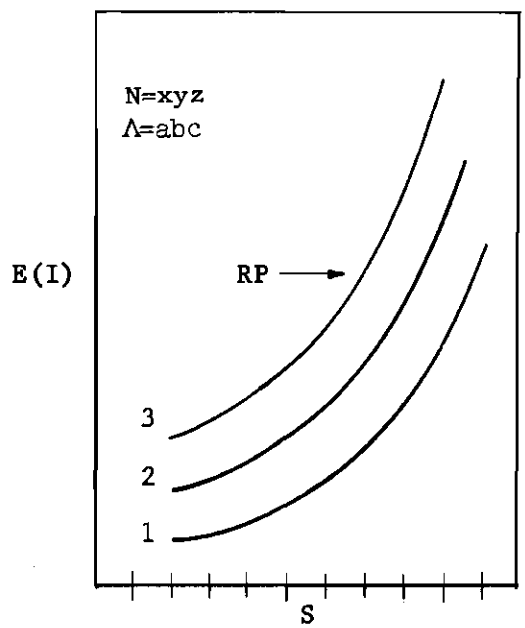

Fig. 34. Suggestive Relationship of E(I) to S for Values of RP with N and Λ Specified for the Combination Inventory Policy

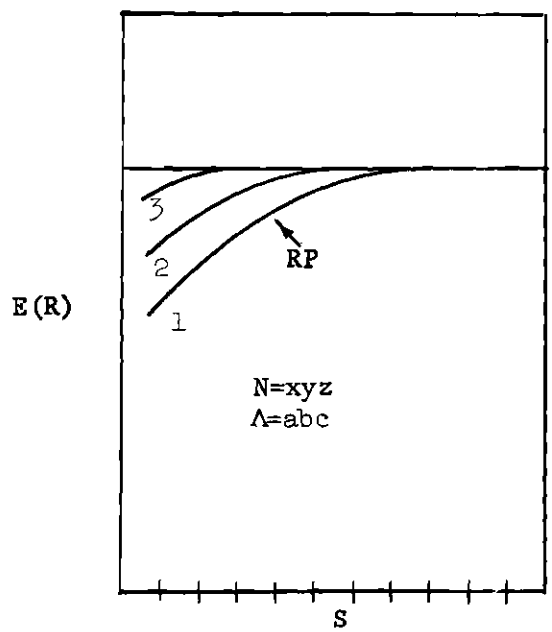

Fig. 35. Suggestive Relationship of E(R) to S for Values of RP with N and Λ Specified for the Combination Inventory Policy

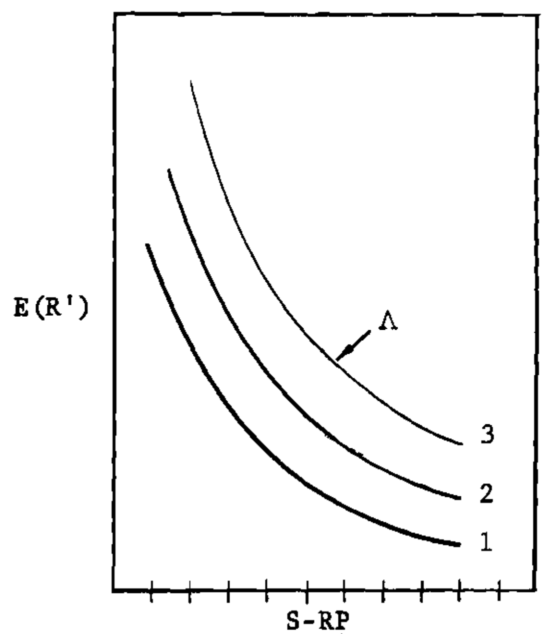

Fig. 36. Suggestive Relationship of E(R') to S-RP for Values of Λ for the Combination Inventory Policy

p_i'	A specified value of the i^{th} decision variable, independent of the decision procedure.
$(\bar{p}_i, \bar{p}_j)$	The value of the i^{th} decision variable and the j^{th} decision variable such that the desired probability of a shortage is most nearly obtained.
$\tilde{p}_i$	The optimal value of the i^{th} decision variable.
i	Decision variable subscript, p_i=S, N, s, RP, Λ.
ϵ	A tolerated probability of a shortage.
α	Inventory policy subscript, α=1 implies the fixed cycle inventory policy, α=2 implies the (s,S) inventory policy, α=3 implies the variable cycle inventory policy, and α=4 implies the combination inventory policy.
β	Index for measure of effectiveness, β = L, T, I, R, R'.
$E_\alpha C(p_1, p_2, \ldots)$	Expected total relevant cost for α^{th} inventory policy for decision variables p_1, p_2, ...
$P_\alpha\{L(p_1, p_2, \ldots)\}$	Probability of a shortage for the α^{th} inventory policy for the decision variables p_1, p_2, ...
$E_\alpha \beta(p_1, p_2, \ldots)$	Expected value of the β^{th} measure of effectiveness for the α^{th} inventory policy for the decision variables p_1, p_2, ...
k_i, k'_i, k''_i	Arbitrary constants.

c_i The i^{th} cost parameter.

θ_i The i^{th} distribution parameter.

Inventory Costs

The objective of this section is to list and define costs relevant to inventory decision procedures. These inventory costs include shortage cost, inventory carrying cost, ordering cost, and surveillance cost. It will be assumed that the material cost of the item is constant. Moreover, it will be assumed that these relevant inventory costs are independent of the passage of time.

Shortage Cost

The cost of a shortage is a loss, both present and future, associated with the failure to satisfy demand immediately. In the present study, the total cost of shortages is considered to be proportional to either (1) the number of shortages or (2) the product of the number of shortages and the duration of the shortages. That is,

$$C_L = c_L(L), \tag{1}$$

or

$$C_L = c_T(T), \tag{2}$$

where C_L = total cost of shortages,

c_L = cost of one shortage without regard to duration of shortage,

c_T = cost of one shortage for one time period,

L = number of shortages random variable, and

T = product of shortages and duration random variable (intensity of shortages).

It is implicit in expressions (1) and (2) that the total cost of shortages is linear either with L or T. Though not necessarily applicable to all practical situations, such an assumption will be used throughout this development.

Carrying Cost

The carrying cost is the consumption of resources associated with having items in stock. This cost usually includes lost interest on alternative investment opportunities, losses attributable to obsolescense, deterioration, spoilage, taxes, insurance, and the cost of storage facilities. In the present study the total carrying cost is considered to be proportional to the number of items in stock. That is,

$$C_I = c_I(I), \tag{3}$$

where C_I = total carrying cost,

c_I = cost of carrying one item for one cycle, and

I = number of items in stock random variable.

It is implicit in expression (3) that the total carrying cost is linear with I. Though not necessarily applicable to all practical situations, such an assumption will be used throughout this development.

Ordering Cost

The cost of ordering is the consumption of resources associated with placing replenishment orders. This study recognizes that the ordering costs for routine replenishment orders (placed at fixed intervals) are not necessarily identical with the ordering costs for special replenishment orders (placed at variable intervals). In the present study, the total ordering cost, for either routine orders or special orders, is considered to be proportional to the number of replenishment orders placed. That is,

$$C_R = c_R(R), \tag{4}$$

and

$$C_{R'} = c_{R'}(R'), \tag{5}$$

where C_R = total routine ordering cost,

$C_{R'}$ = total special ordering cost,

c_R = cost of one routine order,

$c_{R'}$ = cost of one special order,

R = number of routine orders per cycle random variable, and

R' = number of special orders per cycle random variable.

It is implicit in expressions (4) and (5) that the total routine and special ordering cost is linear with R and R', respectively. Though not necessarily applicable to all practical situations, such an assumption will be used throughout this development.

Surveillance Cost

The surveillance cost is the consumption of resources associated with the monitoring of the stock level. For the fixed cycle inventory policy and the (s,S) inventory policy, the stock level is monitored at the beginning of each cycle. For the variable cycle inventory policy and the combination inventory policy, the stock level is usually monitored several times each cycle; the total surveillance cost per order cycle is considered to be proportional to the number of periods in which a replenishment order may be placed. That is, for the variable cycle inventory policy,

$$C_S = c_S(\Omega); \tag{6}$$

and for the combination inventory policy,

$$C_S = c_S(\Lambda\text{-}1). \tag{7}$$

where C_S = total surveillance cost per order cycle,

c_S = surveillance cost for one period,

Ω = length of cycle random variable, and

Λ-1 = periods in which within-the-cycle replenishment orders can be placed.

It is reasonable to assume that the total surveillance cost is linear with the number of periods in which an order may be placed.

Total Relevant Inventory Costs

Of all inventory costs, the cost of a shortage is usually the most difficult to obtain. If the cost of a shortage is known and is proportional to the number of shortages, the equations for the total relevant cost per period for each of the inventory policies are summarized as follows:

Fixed cycle inventory policy and the (s,S) inventory policy.--The total relevant cost per period random variable, C, is

$$C = \frac{1}{N}\,[c_L(L) + c_I(I) + c_R(R) + c_S]. \tag{8}$$

Variable cycle inventory policy.--The total relevant cost per period random variable, C, is

$$C = \frac{1}{\Omega}\,[c_L(L) + c_I(I) + c_{R'}(R') + c_S(\Omega)]. \tag{9}$$

Combination inventory policy.--The total relevant cost per period random variable cost, C, is

$$C = \frac{1}{N}\,[c_L(L) + c_I(I) + c_R(R) + c_{R'}(R') + c_S(\Lambda\text{-}1)]. \tag{10}$$

If the cost of shortages is proportional to the product of the shortages and the duration of the shortages, the terms c_L and L in expressions (8), (9), and (10) are replaced by c_T and T, respectively.

The relevant inventory costs have been listed and defined in this section. These inventory costs will be required in the following section in the development of procedures for the selection of values of the decision variables.

Procedures for the Selection of Optimal* Values of Decision Variables

The objective of this section is to develop procedures for the selection of optimal values of the decision variables for each of the

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*The subject of this footnote is with the existence of optimal values of the decision variables. For practical reasons, there is an upper bound and a lower bound on both the order level, S, and the number of periods, N, in the order cycle. Some of the inventory policies have decision variables in addition to S and N. However, for all of the inventory policies, either S or N is an upper bound for these other decision variables. Also, the lower bound for all decision variables is equal to or greater than zero. Hence, the set of possible optimal values is bounded for all inventory policies. Therefore, a minimum exists and the corresponding values of the decision variables also exist and are optimal. The order level, S, may be bounded for such reasons as space inventory policies studied. The inventory costs defined in this chapter and the measures of effectiveness determined in Chapter VII will be used in this development.

The assumptions under which the four inventory policies were studied do not permit the optimal values of the decision variables to be developed readily as a closed mathematical expression. Also, there is little knowledge relative to the characteristics of total relevant cost as a function of the decision variables. It is possible that multiple local minima occur. Consequently, only general procedures will be developed.

Procedures for the selection of decision variables will be developed for each of the four policies under conditions in which (1) one or more of the decision variables are specified by considerations external to the mathematical model, (2) all of the decision variables are to be jointly specified, and (3) the cost of a shortage is not known, but a tolerated probability of a shortage is specified.

Fixed Cycle Inventory Policy

The expected total relevant cost per period, E_1C, for the fixed cycle policy is obtained by taking the expected value of expression (8) as

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availability restrictions upon the storage of large quantities of items and financial restrictions upon large capital outlays. The number of periods, N, in the order cycle may be bounded for such reasons as large values of N require large values of S or excessive shortages will occur.

$$E_1C = \frac{1}{N}\,[c_L E_1 L + c_I E_1 I + c_R E_1 R + c_S]. \tag{11}$$

For any values of the parameters, say S and N, the expected total relevant cost per period is denoted as

$$E_1C(S,N) = \frac{1}{N}\,[c_L E_1 L(S,N) + c_I E_1 I(S,N) + c_R E_1 R(S,N) + c_S]. \tag{12}$$

If the cost of a shortage is not known and if a tolerated probability of a shortage, ϵ, is specified, the expected total relevant cost per period is denoted as

$$E_1C(\overline{S},\overline{N}|\epsilon) = \frac{1}{N}\,[c_I E_1 I(\overline{S},\overline{N}|\epsilon) + c_R E_1 R(\overline{S},\overline{N}|\epsilon) + c_S]. \tag{13}$$

Selection of order level, $\tilde{S}$, for a given number of periods, N'.--The optimal value for the order level, $\tilde{S}$, given the number of periods, N', between the placement of successive replenishment orders can be determined from the following procedure:

$$\text{Choose } \tilde{S}_{N'} \text{ such that } E_1C(\tilde{S}_{N'},N') = \min_S\{E_1C(S,N')\}. \tag{14}$$

Reference Figures 23, 24, and 25 (page 200).

Joint selection of order level, $\tilde{S}$, and number of periods, $\tilde{N}$.--The optimal values for the order level, $\tilde{S}$, and the number of periods, $\tilde{N}$, between the placement of successive replenishment orders can be determined from the following procedure:

1. Choose $\tilde{S}_N$ such that

$$E_1C(\tilde{S}_N,N) = \min_S\{E_1C(S,N)\} \tag{15}$$

for admissible values of N. Reference Figures 23, 24, and 25. (page 200).

2. Choose $(\tilde{S},\tilde{N})$ such that

$$E_1C(\tilde{S},\tilde{N}) = \min_N\{E_1C(\tilde{S}_N,N)\}. \tag{16}$$

Reference Figures 23, 24, and 25 (page 200).

<u>Selection of order level, $\tilde{S}$, for a given number of periods, N', and a tolerated probability of a shortage, ϵ.</u>--The optimal value for the order level, $\tilde{S}$, given the number of periods, N', between the placement of successive replenishment orders and the tolerated probability of a shortages, ϵ, can be determined from the following procedure:

Choose $\tilde{S}_{N'}$ such that

$$P\{L(\tilde{S},N')\} \doteq \epsilon. \tag{17}$$

Reference Figure 23 * (page 200).

<u>Joint selection of order level, $\tilde{S}$, and number of periods, $\tilde{N}$, given a tolerated probability of a shortage, ϵ.</u>--The optimal values for the order level, $\tilde{S}$, and the number of periods, $\tilde{N}$, between successive replenishment orders, given the tolerated probability of a shortage, ϵ, can be determined from the following procedure:

1. Choose $(\bar{S},\bar{N})$ such that

$$P\{L(\bar{S},\bar{N})\} \doteq \epsilon \tag{18}$$

for all admissible pairs of (S,N). Reference Figure 23 (page 200).

2. Choose $\tilde{S}$, $\tilde{N}$ such that

$$E\ C(S,N|\epsilon) = \min_{(\bar{S},\bar{N})}\{E\ C(\bar{S},\bar{N}|\epsilon)\}. \tag{19}$$

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*The symbol $\doteq$ is used to denote approximately equal.

The (s,S) Inventory Policy

The expected total relevant cost per period, E_2C, for the (s,S) inventory policy is obtained by taking the expected value of expression (8) as

$$E_2C = \frac{1}{N}\left[c_L E_2 L + c_I E_2 I + c_R E_2 R + c_S\right]. \tag{20}$$

For any values of the parameters; say S, s, and N; the expected total relevant cost is denoted as

$$E_2C(S,s,N) = \frac{1}{N}\left[c_L E_2 L(S,s,N) + c_I E_2 I(S,s,N) + c_R E_2 R(S,s,N) + c_S\right]. \tag{21}$$

If the cost of a shortage is not known and if a tolerated probability of a shortage is specified, the expected total relevant cost per period is denoted as

$$E_2C(\bar{S},\bar{s},N|\epsilon) = \frac{1}{N}\left[c_I E_2 I(\bar{S},\bar{s},N|\epsilon) + c_R E_2 R(\bar{S},\bar{s},N|\epsilon) + c_S\right]. \tag{22}$$

Selection of order levels, $\tilde{S}$ and $\tilde{s}$, for a given number of periods, N'.--The optimal values for the order levels, $\tilde{S}$ and $\tilde{s}$, given the number of periods, N', between the placement of successive replenishment orders can be determined from the following procedure:

1. Choose $\tilde{S}_{s,N'}$ such that

$$E_2C(\tilde{S}_{s,N'}, s, N') = \min_S\{E_2C(S,s,N')\} \tag{23}$$

for all admissible values of s. Reference Figures 26, 27, and 28.

2. Choose $(\tilde{S}_{N'}, \tilde{s}_{N'})$ such that

$$E_2C(\tilde{S}_{N'}, \tilde{s}_{N'}, N') = \min_s\{E_2C(\tilde{S}_{s,N'}, s, N')\}. \tag{24}$$

Reference Figures 26, 27, and 28 (pages 200 and 202).

Joint selection of order levels, $\tilde{S}$ and $\tilde{s}$, and the number of periods, $\tilde{N}$.--The optimal values for the order levels, $\tilde{S}$ and $\tilde{s}$, and the number of periods, $\tilde{N}$, between the placement of successive replenishment orders can be determined from the following procedure:

1. Choose $\tilde{S}_{s,N}$ such that

$$E_2C(\tilde{S}_{s,N}, s, N) = \min_{S}\{E_2C(S, s, N)\} \tag{25}$$

for all admissible pairs of (s,N). Reference Figures 26, 27, and 28.

2. Choose $(\tilde{S}_N, \tilde{s}_N)$ such that

$$E_2C(\tilde{S}_N, \tilde{s}_N, N) = \min_{s}\{E_2C(\tilde{S}_{s,N}, s, N)\} \tag{26}$$

for all admissible values of N. Reference Figures 26, 27, and 28.

3. Choose $(\tilde{S}, \tilde{s}, \tilde{N})$ such that

$$E_2C(\tilde{S}, \tilde{s}, \tilde{N}) = \min_{N}\{E_2C(\tilde{S}_N, \tilde{s}_N, N)\}. \tag{27}$$

Reference Figures 26, 27, and 28 (pages 200 and 202).

Selection of order levels, $\tilde{S}$ and $\tilde{s}$, for a given number of periods, N', and a tolerated probability of a shortage, ϵ.--The optimal values for the order levels, $\tilde{S}$ and $\tilde{s}$, given the number of periods, N', between the placement of successive replenishment orders and the tolerated probability of a shortage, ϵ, can be determined from the following procedure:

1. Choose $(\bar{S}_{N'}, \bar{s}_{N'})$ such that

$$P\{L(\bar{S}_{N'}, \bar{s}_{N'})\} \doteq \epsilon \tag{28}$$

for all admissible pairs of (s,S). Reference Figure 26 (page 200).

2. Choose $(\tilde{S}_{N'}, \tilde{s}_{N'})$ such that

$$E_2C(\tilde{S}_{N'}, \tilde{s}_{N'}, N'|\epsilon) = \min_{(\bar{S}, \bar{s})} \{E_2C(\bar{S}_{N'}, \bar{s}_{N'}, N'|\epsilon)\}. \tag{29}$$

Reference Figures 26, 27, and 28 (pages 200 and 202).

Selection of order levels, $\tilde{S}$ and $\tilde{s}$, and number of periods, $\tilde{N}$, given a tolerated probability of a shortage, ϵ.--The optimal values for the order levels, $\tilde{S}$ and $\tilde{s}$, and the number of periods, $\tilde{N}$, between successive replenishment orders, given the tolerated probability of a shortage, ϵ, can be determined from the following procedure:

1. Choose $(\bar{S}_N, \bar{s}_N)$ such that

$$P\{I(\bar{S}, \bar{s}, N)\} \doteq \epsilon \tag{30}$$

for all admissible pairs of (s,S) and for all admissible values of N.

2. Choose $(\tilde{S}_N, \tilde{s}_N)$ such that

$$E_2C(\tilde{S}, \tilde{s}, N) = \min_{(\bar{s}, \bar{S})} \{E_2C(\bar{S}_N, \bar{s}_N, N)\} \tag{31}$$

for all admissible values of N. Reference Figures 26, 27, and 28.

3. Choose $(\tilde{S}, \tilde{s}, \tilde{N})$ such that

$$E_2C(\tilde{S}, \tilde{s}, \tilde{N}) = \min_{N} \{E_2C(\tilde{S}_N, \tilde{s}_N, N)\}. \tag{32}$$

Reference Figures 26, 27, and 28 (pages 200 and 202).

Variable Cycle Inventory Policy

The expected total relevant cost per period E_3C, for the variable cycle inventory policy, is obtained by taking the expected value of expression (9) as

$$E_3C = \frac{1}{E_3\Omega}\left[c_L E_3 L + c_I E_3 I + c_{R'} E_3 R' + c_S E_3 \Omega \right].$$

For any values of the parameters, say S and RP, the expected total relevant cost per period is denoted as

$$E_3C(S,RP) = \frac{1}{E_3\Omega(S,RP)}\left[c_L E_3 L(S,RP) + c_I E_3 I(S,RP) + c_{R'} E_3 R'(S,RP) + c_S E_3 \Omega(S,RP) \right]. \qquad (34)$$

When the cost of a shortage is not known and a tolerated probability of a shortage is specified, the expected total relevant cost per period is denoted as

$$E_3C(\overline{S},\overline{RP}|\epsilon) = \frac{1}{E_3\Omega(\overline{S},\overline{RP}|\epsilon)}\left[c_I E_3 I(\overline{S},\overline{RP}|\epsilon) + c_{R'} E_3 R'(\overline{S},\overline{RP}|\epsilon) + c_S E_3 \Omega(\overline{S},\overline{RP}|\epsilon) \right]. \qquad (35)$$

<u>Selection of order level, $\tilde{S}$, and reorder point, $\tilde{RP}$</u>.--The optimal values for the order level, $\tilde{S}$, and the reorder point, $\tilde{RP}$, can be determined by the following procedure:

1. Choose $\tilde{S}_{RP}$ such that

$$E_3C(\tilde{S}_{RP},RP) = \min_{S} \{E_3C(S,RP)\} \qquad (36)$$

for all admissible values of RP. Reference Figures 29, 30, 31, and 32 (pages 202 and 203).

2. Choose $(\tilde{S},\tilde{RP})$ such that

$$E_3C(\tilde{S},\tilde{RP}) = \min_{RP} \{E_3C(\tilde{S}_{RP},RP)\}. \qquad (37)$$

Reference Figures 29, 30, 31, and 32 (pages 202 and 203).

<u>Selection of order level, $\tilde{S}$, and reorder point, $\widetilde{RP}$, given a tolerated probability of a shortage, ϵ</u>.--The optimal values for the order level, $\tilde{S}$, and the reorder point, $\widetilde{RP}$, given the tolerated probability of a shortage, ϵ, can be determined from the following procedure:

1. Choose $(\overline{S},\overline{RP})$ such that

$$P_3\{L(\overline{S},\overline{RP})\} \doteq \epsilon \tag{38}$$

for all admissible pairs of $(\overline{RP},\overline{S})$. Reference Figure 29 (page 202).

2. Choose $(\tilde{S},\widetilde{RP})$ such that

$$E_3C(\tilde{S},\widetilde{RP}|\epsilon) = \min_{(\overline{RP},\overline{S})} \{E_3C(\overline{S},\overline{RP}\ |\epsilon)\} \ . \tag{39}$$

Reference Figures 29, 30, 31, and 32 (pages 202 and 203).

Combination Inventory Policy

The expected total relevant cost per period, E_4C, for the combination inventory policy, is obtained by taking the expected value of expression (10) as

$$E_4C = \frac{1}{N}\ [c_LE_4L + c_IE_4I + c_RE_4R + c_{R'}E_4R' + c_S(\Lambda-1)]. \tag{40}$$

For any of the values of the parameters; say S, N, RP, and Λ; the expected total relevant cost per period is denoted as

$$E_4C(S,N,RP,\Lambda) = \frac{1}{N}\ [c_L E_4 L(S,N,RP,\Lambda) + c_I E_4 I(S,N,RP,\Lambda) + c_R E_4 R(S,N,RP,\Lambda) + c_{R'} E_4 R'(S,N,RP,\Lambda) + c_S(\Lambda-1)]. \tag{41}$$

When the cost of a shortage is not known and a tolerated probability of a shortage is specified, the expected total relevant cost is denoted as

$$E_4C(\overline{S},\overline{RP},N,\Lambda|\epsilon) = \frac{1}{N}\ [c_I E_4 I(\overline{S},\overline{RP},N,\Lambda|\epsilon) + c_R E_4 R(\overline{S},\overline{RP},N,\Lambda|\epsilon) + c_{R'} E_4 R'(\overline{S},\overline{RP},N,\Lambda|\epsilon) + c_S(\Lambda-1)]. \tag{42}$$

<u>Joint selection of order level, $\tilde{S}$, reorder point, $\widetilde{RP}$, given special ordering during first Λ' periods, and the number of periods, N'</u>.-- The optimal values for the order level, $\tilde{S}$, and the reorder point, $\widetilde{RP}$, given the number of periods for special ordering, Λ', and the number of periods in the order cycle, N', can be determined by the following procedure:

1. Choose $\tilde{S}_{RP,\Lambda',N'}$ such that

$$E_4[c(\tilde{S}_{RP,\Lambda',N'}, RP,\Lambda',N')] = \min_S\{E_4[C(S,RP,\Lambda',N')]\} \tag{43}$$

for all admissible values of RP. Reference Figures 33, 34, 35, and 36.

2. Choose $(\tilde{S}_{\Lambda',N'},\widetilde{RP}_{\Lambda',N})$ such that

$$E_4[C(\tilde{S}_{\Lambda',N'},\widetilde{RP}_{\Lambda',N'})] = \min_{RP}\{E_4[C(\tilde{S}_{RP,\Lambda',N'},RP,\Lambda',N')]. \tag{44}$$

Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

<u>Joint selection of order level, $\tilde{S}$, reorder point, $\widetilde{RP}$, and for special ordering during first $\tilde{\Lambda}$ periods, given the number of periods, N'</u>.--The optimal values for the order level, $\tilde{S}$, the reorder point, $\widetilde{RP}$, and the number of periods $\tilde{\Lambda}$ for special ordering, given the number of periods, N', will be determined from the following procedure:

1. Choose $\tilde{S}_{RP,\Lambda,N'}$ such that

$$E_4C(\tilde{S}_{RP,\Lambda,N'},RP,\Lambda,N') = \min_{S} E_4C(S,RP,\Lambda,N') \tag{45}$$

for all admissible paris of (RP,Λ) for a given N'. Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

2. Choose $(\tilde{S}_{\Lambda,N'},\widetilde{RP}_{\Lambda,N'})$ such that

$$E_4C(\tilde{S}_{\Lambda,N'},\widetilde{RP}_{\Lambda,N'},\Lambda,N') = \min_{RP}\{E_4C(\tilde{S}_{RP,\Lambda,N'},RP,\Lambda,N')\} \tag{46}$$

for all admissible values of Λ. Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

3. Choose $(\tilde{S}_{N'},\widetilde{RP}_{N'},\tilde{\Lambda}_{N'})$ such that

$$E_4C(\tilde{S}_{N'},\widetilde{RP}_{N'},\tilde{\Lambda}_{N'},N') = \min_{\Lambda}\{E_4C(\tilde{S}_{\Lambda,N'},\widetilde{RP}_{\Lambda,N'},\Lambda,N')\}. \tag{47}$$

Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

<u>Joint selection of order level, $\tilde{S}$, reorder point, $\widetilde{RP}$, the number of periods, $\tilde{N}$, and for special ordering during first $\tilde{\Lambda}$ periods</u>.--The optimal values for the order level, $\tilde{S}$, the reorder point, $\widetilde{RP}$, the number of periods, $\tilde{N}$, and the number of periods, $\tilde{\Lambda}$, in which a special order may be placed, can be determined from the following procedure:

1. Choose $\tilde{S}_{N,RP,\Lambda}$ such that

$$E_4C(\tilde{S}_{N,RP,\Lambda}, N, RP, \Lambda) = \min E_4C(S, N, RP, \Lambda) \tag{48}$$

for all admissible triplets of N, RP, and Λ.

2. Choose $(\tilde{S}_{RP,\Lambda}, \tilde{N}_{RP,\Lambda})$ such that

$$E_4C(\tilde{S}_{RP,\Lambda}, \tilde{N}_{RP,\Lambda}, RP, \Lambda) = \min_N \{E_4C(\tilde{S}_{N,RP,\Lambda}, N, RP, \Lambda)\} \tag{49}$$

for all admissible pairs of RP and Λ. Reference Figures 33, 34, 35, and 36.

3. Choose $(\tilde{S}_\Lambda, \tilde{N}_\Lambda, \tilde{RP}_\Lambda)$ such that

$$E_4C(\tilde{S}_\Lambda, \tilde{N}_\Lambda, \tilde{RP}_\Lambda, \Lambda) = \min_{RP} \{E_4C(\tilde{S}_{RP,\Lambda}, N_{RP,\Lambda}, RP, \Lambda)\} \tag{50}$$

for all admissible values of Λ. Reference Figures 33, 34, 35, and 36.

4. Choose $(\tilde{S}, \tilde{N}, \tilde{RP}, \tilde{\Lambda})$ such that

$$E_4C(\tilde{S}, \tilde{N}, \tilde{RP}, \tilde{\Lambda}) = \min_\Lambda \{E_4C(\tilde{S}_\Lambda, \tilde{N}_\Lambda, \tilde{RP}_\Lambda, \Lambda)\}. \tag{51}$$

Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

<u>Joint selection of order level, $\tilde{S}$, and reorder point, $\tilde{RP}$, given special ordering during first Λ' periods, the number of periods, N', and given a tolerated probability of a shortage, ϵ.</u>--The optimal values for the order level, $\tilde{S}$, and the reorder point, $\tilde{RP}$, given the number of periods for special ordering, Λ', the number of periods in the order cycle, N', and the tolerated probability of a shortage, ϵ, can be determined from the following procedure:

1. Choose $(\overline{S}_{\Lambda',N'}, \overline{RP}_{\Lambda',N'})$ such that

$$P_4\{L; \overline{S}_{\Lambda',N'}, \overline{RP}_{\Lambda',N'}, \Lambda', N'\} \doteq \epsilon \tag{52}$$

for all admissible pairs of $(\overline{S}, \overline{RP})$. Reference Figure 33.

2. Choose $(\tilde{S}_{\Lambda', N'}, \tilde{RP}_{\Lambda', N'})$ such that

$$E_4 C(\tilde{S}_{\Lambda', N'}, \tilde{RP}_{\Lambda', N'}, \Lambda', N' \mid \epsilon) = \min_{(\overline{S}, \overline{RP})} \{E_4 C(\overline{S}_{\Lambda', N'}, \overline{RP}_{\Lambda', N'}, \Lambda', N')\} \tag{53}$$

Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

<u>Joint selection of order level, $\tilde{S}$, reorder point, $\tilde{RP}$, and for special ordering during first $\tilde{\Lambda}$ periods, given the number of periods, N', and given a tolerated probability of a shortage, ϵ</u>.--The optimal values for the order level, $\tilde{S}$, the reorder point, $\tilde{RP}$, and the number of periods $\tilde{\Lambda}$ for special ordering, given the number of periods, N', and the tolerated probability of a shortage, ϵ, can be determined from the following procedure:

1. Choose $(\overline{S}_{\Lambda, N'}, \overline{RP}_{\Lambda, N'})$ such that

$$P_4\{L(\overline{S}_{\Lambda, N'}, \overline{RP}_{\Lambda, N'}, \Lambda, N'\} \doteq \epsilon \tag{54}$$

for all admissible pairs of $(\overline{S}, \overline{RP})$ for each admissible value of Λ. Reference Figure 33 (page 203).

2. Choose $(\tilde{S}_{\Lambda, N'}, \tilde{RP}_{\Lambda, N'})$ such that

$$E_4 C(\tilde{S}_{\Lambda, N'}, \tilde{RP}_{\Lambda, N'}, \Lambda, N' \mid \epsilon)$$

$$= \min_{(\overline{S}, \overline{RP})} \{E_4 C(\overline{S}_{\Lambda, N'}, \overline{RP}_{\Lambda, N'}, \Lambda, N' \mid \epsilon)\} \tag{55}$$

for admissible values of Λ. Reference Figures 33, 34, 35, and 36.

3. Choose $(\tilde{S}_{N'}, \tilde{RP}_{N'}, \tilde{\Lambda}_{N'})$ such that

$$E_4C(\tilde{S}_{N'},\tilde{RP}_{N'},\tilde{\Lambda}_{N'},N'|\epsilon) = \min_{\Lambda} E_4C(\tilde{S}_{\Lambda,N'},\tilde{RP}_{\Lambda,N'},\Lambda,N'|\epsilon). \quad (56)$$

Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

Joint selection of order level, $\tilde{S}$, reorder point, $\tilde{RP}$, the number of periods, $\tilde{N}$, and for special ordering during first $\tilde{\Lambda}$ periods, given a tolerated probability of a shortage, ϵ.--The optimal values for the order level, $\tilde{S}$, the reorder point, $\tilde{RP}$, the number of periods, $\tilde{N}$, and the number of periods, $\tilde{\Lambda}$, in which a special order may be placed, given a tolerated probability of a shortage, ϵ, can be determined from the following procedure:

1. Choose $(\overline{S}_{\Lambda,N},\overline{RP}_{\Lambda,N})$ such that

$$P_4\{L(\overline{S}_{\Lambda,N},\overline{RP}_{\Lambda,N},\Lambda,N)\} \doteq \epsilon \quad (57)$$

for all admissible pairs of $(\overline{S},RP)$ for each admissible pair of (Λ,N). Reference Figure 33 (page 203).

2. Choose $(\tilde{S}_{\Lambda,N},\tilde{RP}_{\Lambda,N})$ such that

$$E_4C(\tilde{S}_{\Lambda,N},\tilde{RP}_{\Lambda,N},\Lambda,N|\epsilon) = \min_{(\overline{S},\overline{RP})}\{E_4C(\overline{S}_{\Lambda,N},\overline{RP}_{\Lambda,N},\Lambda,N|\epsilon)\} \quad (58)$$

for all admissible pairs of (Λ,N). Reference Figures 33, 34, 35, and 36.

3. Choose $(\tilde{S}_{\Lambda},\tilde{RP}_{\Lambda},\tilde{N}_{\Lambda})$ such that

$$E_4C(\tilde{S}_{\Lambda},\tilde{RP}_{\Lambda},N_{\Lambda},\Lambda|\epsilon) = \min_{N}\{E_4C(\tilde{S}_{\Lambda,N},\tilde{RP}_{\Lambda,N},\Lambda,N|\epsilon)\} \quad (59)$$

for all admissible values of Λ. Reference Figures 33, 34, 35, and 36.

4. Choose $(\tilde{S},\tilde{RP},\tilde{N},\tilde{\Lambda})$ such that

$$E_4C(\tilde{S},\tilde{RP},\tilde{N},\tilde{\Lambda}|\epsilon) = \min_{\Lambda}\{E_4C(\tilde{S}_{\Lambda},\tilde{RP}_{\Lambda},N_{\Lambda},\Lambda|\epsilon)\}. \quad (60)$$

Reference Figures 33, 34, 35, and 36 (pages 203 and 204).

Procedures for the selection of optimal values of the decision variables for each of the inventory policies studied have been developed. This development satisfies the first of the secondary objectives of this study. The following two sections utilize optimal values of the decision variables.

Procedures for the Study of the Sensitivity of Total Relevant Cost

The objective of this section is to develop procedures for the study of the sensitivity of total relevant cost. Rather than being applicable to a specific inventory policy, these procedures will be general. Procedures will be developed for (1) the study of the effects upon total relevant cost of deviations in the values of decision variables from optimal values and (2) the study of the effects upon total relevant cost of errors in the estimation of cost and distribution parameters.

Sensitivity curves will be developed as a part of the procedure for the study of parameter sensitivity. For both (1) and (2) above, the ordinate of the sentivity curves will be the ratio of the non-optimal total relevant cost C, to the optimal relevant cost, $\tilde{C}$. For (1) above, the abscissa of the sensitivity curves will be the ratio of non-optimal value of the decision variable, p_i, to the optimal value of the decision variable, $\tilde{p}_i$. For (2) above, the abscissa of the sensitivity curves will be either the ratio of the estimated value of the i^{th} cost parameter $\bar{c}_i$, to the true value of the i^{th} cost parameter, c_i, or the ratio of the estimated value of the i^{th} distribution parameter, $\bar{\theta}_i$, to the true value of the i^{th} distribution parameter, θ_i.

Throughout this section it will be assumed that the optimal values of the decision variables can be selected by the procedures developed in the preceding section.

Deviations in Values of Decision Variables

The total relevant cost varies with the values of the decision variables. A procedure will be developed which will yield information regarding the precision required in the selection of values of the decision variables.

Some inventory situations may have inventory costs of a nature such that minor deviations in values of the decision variables result in major deviations in the total relevant cost. And conversely, some inventory situations may have inventory costs of a nature such that major deviations in values of the decision variables result in minor deviations in the total relevant cost. Consequently, as a result of the interrelationships of inventory costs, the application of procedures for the selection of values of the decision variables will not always be equally rewarding. In some cases, crude values of the decision variables may be satisfactory. Therefore, the use of elaborate procedures for the selection of values of the decision variables may not be justified. In other cases, precise values of the decision variables may be necessary.

The procedure for developing sensitivity curves is as follows:

1. Select the decision variable of interest, p_i.
2. Choose $\tilde{p}_1$, $\tilde{p}_2$, . . . such that $E_\alpha C(\tilde{p}_1, \tilde{p}_2, \ldots)$ is minimum. $E_\alpha C(\tilde{p}_1, \tilde{p}_2, \ldots)$ will be denoted as $\tilde{C}$; where α is inventory policy subscript.
3. Select values for all decision variables other than p_i;

in the following the k_i are arbitrary constants.
That is, $p_1 = k_1\tilde{p}_1$, ..., $p_{i-1} = k_{i-1}\tilde{p}_{i-1}$, $p_{i+1} = k_{i+1}\tilde{p}_{i+1}$, ...

4. Compute $C_{p_i} = E_\alpha C(p_1, p_2, \ldots)$ for all relevant values of p_i.
5. Plot $Cp_i/\tilde{C}$ versus $p_i/\tilde{p}_i$ for all relevant values of p_i as in Figure 37 (page 227).
6. Repeat steps 2 through 5 for other values of the inventory costs. The j^{th} set of the inventory costs is denoted by the vector $\underline{c}^{(j)} = (c_{1j}, c_{2j}, \ldots)$.

Errors in Estimation of Cost and Distribution Parameters

If values of the decision variables are determined by use of estimated cost and distribution parameters, rather than from knowledge of the true values, the total relevant cost varies with the difference between the estimated values and the true values. A procedure will be developed which will yield information regarding the precision required in the estimation of the cost and distribution parameters. As noted in the previous section, some inventory situations may have inventory costs of a nature such that minor deviations between the estimated values and the true values of the cost and distribution parameters result in major deviations of the total relevant cost, and conversely. Consequently, as a result of the interrelationship of the inventory costs, the application of the procedures for the selection of the values of the decision variables will not always be equally rewarding. In some cases, crude values of cost and distribution parameters may be satisfactory; while in other cases, precise values of the cost and distribution parameters may be necessary.

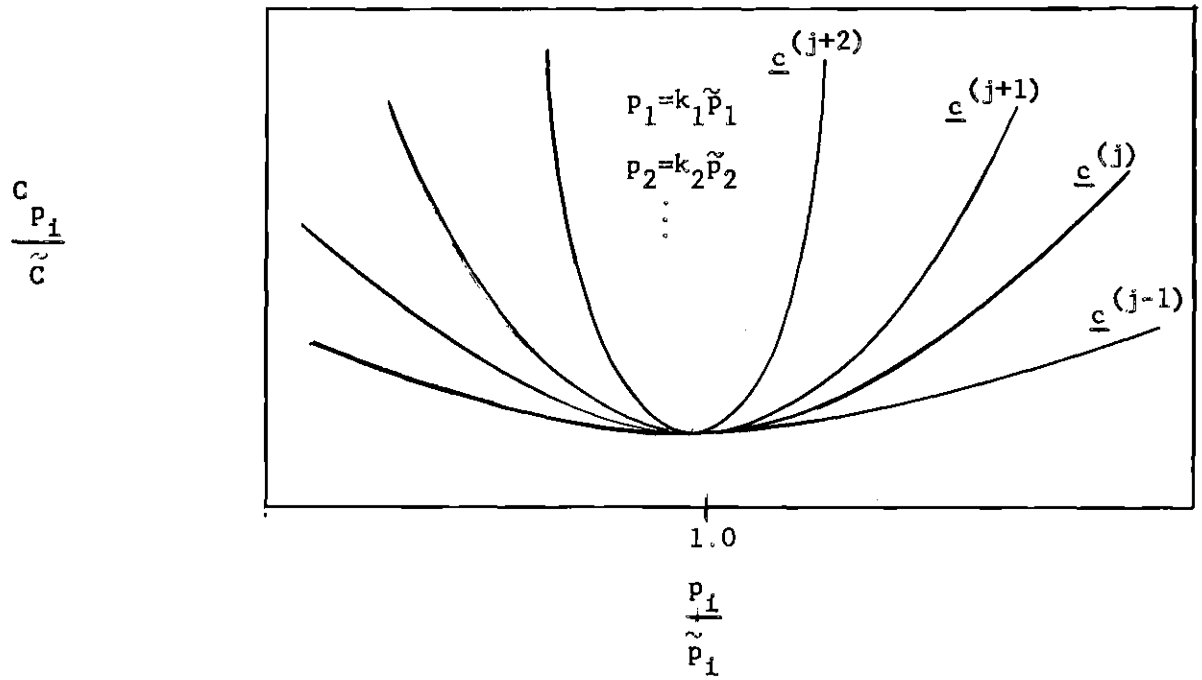

Fig. 37. Sensitivity of Total Relevant Cost to Deviations in Decision Systems

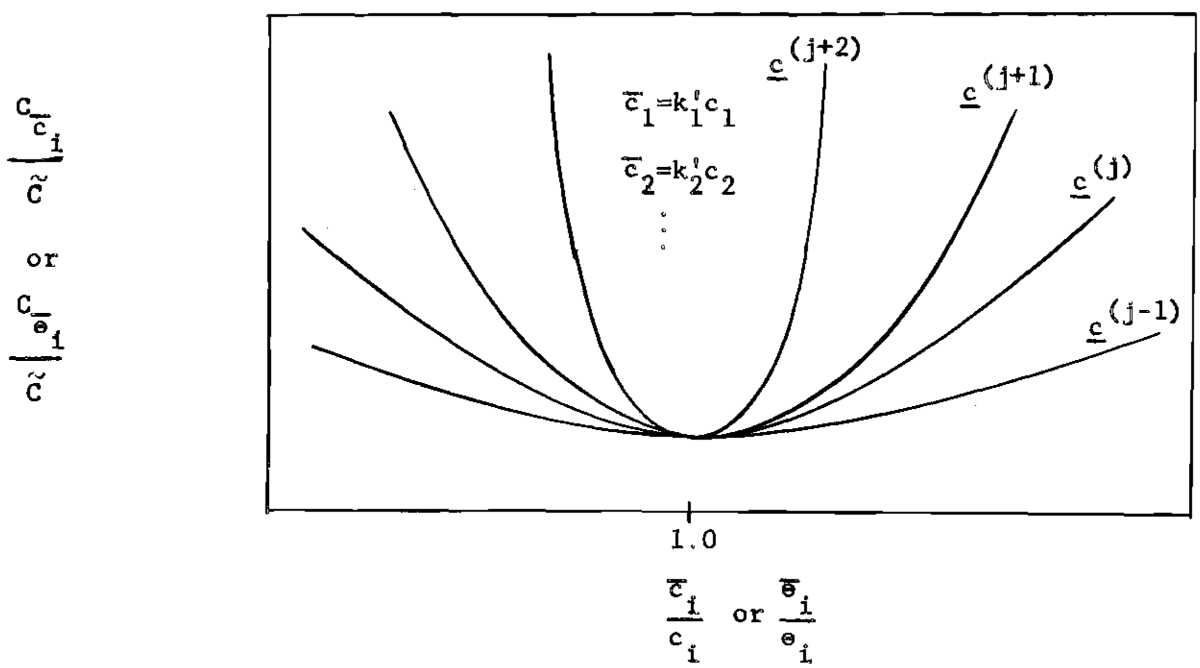

Fig. 38. Sensitivity of Total Relevant Cost to Errors in Estimation of Cost of Distribution Parameters

The procedure for developing the sensitivity curves is as follows:

1. Select the cost or distribution parameter of interest, c_i or θ_i.
2. Select the estimated vaues of all cost or distribution parameters other than c_i or θ_i, $\bar{c}_1 = k'_1 c_1$, ..., $\bar{c}_{i-1} = k'_{i-1} c_{i-1}$, $\bar{c}_{i+1} = k'_{i+1} c_{i+1}$, ...; $\bar{\theta}_1 = k''_1 \theta_1$, ..., $\bar{\theta}_{i-1} = k''_{i-1} \theta_{i-1}$, $\bar{\theta}_{i+2} = k''_{i+2} \theta_{i+2}$, ... The k'_i and k''_i are arbitrary constants.
3. Choose $\tilde{p}_1$, $\tilde{p}_2$, ... such that $EC(\tilde{p}_1, \tilde{p}_2, \ldots)$ is minimum. $EC(\tilde{p}_1, \tilde{p}_2, \ldots)$ will be denoted as $\tilde{C}$.
4. Utilizing the estimated cost and distribution parameters, compute $\tilde{\tilde{p}}_1$, $\tilde{\tilde{p}}_2$, ... such that $E[C(\tilde{\tilde{p}}_1, \tilde{\tilde{p}}_2, \ldots)]$ is minimum for all values of $\bar{c}_i$ or $\bar{\theta}_i$.
5. Compute $C_{\bar{c}_i}$ or $C_{\bar{\theta}_i} = EC(\tilde{\tilde{p}}_1, \tilde{\tilde{p}}_2, \ldots)$ for all values of $\bar{c}_i$ or $\bar{\theta}_i$.
6. Plot $C_{\bar{c}_i}/\tilde{C}$ versus $\bar{c}_i/c_i$ for all values of $\bar{c}_i$, or plot $C_{\bar{\theta}_i}/\tilde{C}$ versus $\bar{\theta}_i/\theta_i$ for all values of $\bar{\theta}_i$ as in Figure 38 (page 227).
7. Repeat steps 2 through 6 for other values of the inventory costs. The j^{th} set of the inventory costs is denoted by the vector $\underline{c}^{(j)} = (c_{1j}, c_{2j}, \ldots)$.

Procedures for the study of the sensitivity of total relevant cost have been developed. This development satisfies the second of the secondary objectives of this study. The study of the sensitivity of total relevant cost is potentially useful.

Procedure for Choosing the Best Policy

The following procedure can be used for determining the best inventory policy from among the four policies studied. It will be assumed that the optimal values of the decision variables and the total relevant cost have been selected as outlined earlier in this chapter.

Choose α such that

$$E_{\alpha}C(\tilde{p}_1, \tilde{p}_2, \ldots) = \min_{\alpha}\{E_{\alpha}C(\tilde{p}_1, \tilde{p}_2, \ldots)\}, \tag{61}$$

where $\alpha=1$ implies the fixed cycle inventory policy,

$\alpha=2$ implies the (s,S) inventory policy,

$\alpha=3$ implies the variable cycle inventory policy, and

$\alpha=4$ implies the combination inventory policy.

The development of this procedure for choosing the best policy satisfied the last of the secondary objectives of this study.

Results

Procedures for the selection of optimal values of decision variables for a given inventory policy, the study of the sensitivity of total relevant cost, and the choosing of the best policy from among the four selected policies have been developed. The development of these procedures satisfies the secondary and final objective of this study. The subsequent chapter, Chapter IX, will state conclusions obtained throughout this study and will recommend further studies in this area.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

1. Significant theoretical developments relevant to the objectives of this study have been summarized in Tables 1, 2, and 3.

2. Expressions for stock level probabilities for the fixed cycle inventory policy have been developed in Chapter III under the hypothesis of back-orders allowed and the hypothesis of back-orders not allowed.

3. Expressions for stock level probabilities for the (s,S) inventory policy have been developed in Chapter IV under the hypothesis of back-orders allowed and the hypothesis of back-orders not allowed.

4. Expressions for length-of-cycle probabilities and stock level probabilities for the variable cycle inventory policy have been developed in Chapter V under the hypothesis of back-orders allowed and the hypothesis of back-orders not allowed.

5. Expressions for stock level probabilities for the combination inventory policy have been developed in Chapter VI under the hypothesis of back-orders allowed.

6. Expressions for the measures of effectiveness have been determined in Chapter VII for the fixed cycle inventory policy, the (s,S) inventory policy, the variable cycle inventory policy, and the combination inventory policy. These measures of effectiveness are the probability

of one or more shortages, the expected number of shortages, the expected intensity of shortages, the expected inventory, and the expected number of replenishment orders.

7. Procedures for the selection of optimal values of decision variables for a given inventory policy, the study of the sensitivity of total relevant cost, and the choosing of the best policy from among the four selected policies have been developed in Chapter VIII.

Recommendations

The following recommendations for additional studies which have been generated during this research are categorized into those of a practical nature and into those of a theoretical nature:

Recommendations of a Practical Nature

1. Studies should be undertaken to ascertain which theoretical probability distributions are typical of the demand and the replenishment lead time distributions which occur in practice.

2. Sets of charts should be developed from the expressions determined in this study for theoretical demand and replenishment lead time distributions.

3. Sets of charts for the sensitivity analysis of total relevant cost should be developed utilizing the procedures developed in this study.

4. Approximation expressions should be developed for each of the precise expressions developed in this study, along with the range for which the approximation is reasonable.

Recommendations of a Theoretical Nature

1. Stock level probabilities should be developed for the combination inventory policy under the hypothesis of back-orders not allowed.

2. The probability of one or more shortages should be determined for the combination inventory policy under the hypothesis of back-orders allowed and also under the hypothesis of back-orders not allowed.

3. Expressions for the inventory policies considered in this study should be developed without an upper bound on replenishment lead time. Also, expressions for the variable cycle inventory policy should be developed without an upper bound on demand during any given period.

4. Stock level probabilities should be developed for each of the inventory policies when a fraction of the customers allow back-orders while the remaining customers do not allow back-orders.

5. Stock level probabilities should be developed for each of the inventory policies when replenishment orders are received in partial shipments.

6. Stock level probabilities should be developed for each of the inventory policies when the replenishment order is subject to a screening process in which defective items are removed.

This research culminates with a conceptual framework in which decision procedures may be facilitated. If and when such a conceptual framework is transformed to a numerical basis, the results of this research can have engineering application.

APPENDIX

Glossary of Symbols

The symbols which are used consistently in several sections are listed here. The Roman numerals in parentheses refer to the chapters in which the symbol is used.

$\underline{a}$	The stationary row vector which has elements, a(i). (III, IV, V, and VI)
$a(i)$	Stationary probability that the stock level at the beginning of the order cycle is equal to i. (III, IV, V, VI, and VII)
$a(i\|\bar{A})$	Conditional stationary probability that the beginning stock level is equal to i, given the set $\bar{A}$. (VI)
$a(i\|AB)$	Conditional stationary probability that the beginning stock level is equal to i, given the set AB. (VI)
$a(i\|A\bar{B})$	Conditional stationary probability that the beginning stock level is equal to i, given the set $A\bar{B}$. (VI)
$a(i\|A\bar{B},\bar{r})$	Conditional stationary probability that the beginning stock level is equal to i, given the set $A\bar{B},\bar{r}$. (VI)
$a_r(h)$	Stationary probability that the stock level prior to the demand in period r is equal to h. (III, IV, V, VI, and VII)
$a_r(h\|x)$	Conditional stationary probability that the stock level prior to the demand in period r is equal to h, given that the replenishment lead time is X=x. (III, IV, and VI)
$a_{r1}(h)$	Joint stationary probability that the stock level prior to the demand in period r is equal to h and that the beginning stock level is i, for $s < i \leq S$. (IV)
$a_{r2}(h)$	Joint stationary probability that the stock level prior to the demand in period r is equal to h and that the beginning stock level is i, for $i \leq s$. (IV)

$a_r(h\|\omega)$	Conditional stationary probability that the stock level prior to the demand in period r is equal to h, given that the length of cycle is $\Omega=\omega$. (V and VII)
$a_r(h\|\omega,x)$	Conditional stationary probability that the stock level prior to the demand in period r is equal to h, given that the length of cycle is $\Omega=\omega$ and that the replenishment lead time is X=x. (V)
$a_r(h\|A\bar{B},x,\bar{r})$	Conditional stationary probability that the stock level prior to the demand in period r is equal to h, given the set $A\bar{B}$, given the replenishment lead time is X=x, and given that the carry-over replenishment order is received at the beginning of period $\bar{r}$. (VI)
$D(\lambda)$	Demand during λ periods random variable. (III, IV, V, VI, and VII)
δ_{hS}	The Kroneker delta symbol which is unity if h=S and zero if $h \neq S$. (III, IV, V, and VI)
E(I)	Expected inventory during ordering cycle. (VII and VIII)
E(L)	Expected number of shortages during the order cycle. (VII and VIII)
$E(\Omega)$	Expected length of cycle. (V, VII, and VIII)
E(R)	Expected number of routine orders placed during order cycle. (VII and VIII)
E(R')	Expected number of special orders placed during order cycle. (VII and VIII)
E(T)	Expected intensity of shortages during the order cycle. (VII and VIII)
h	Index used to indicate value of stock level prior to demand during a period. (III, IV, V, VI, and VII)
i	Index used to indicate value of beginning stock level. (III, IV, V, VI, and VII)
j	Index used in development of the Markov transition matrix; corresponds to a possible value for the stock level at the beginning of the subsequent cycle. (III, IV, V, and VI)
K	Upper bound for replenishment lead time. (V and VII)

Λ	Number of periods in which a within-the-cycle replenishment order may be placed. (VI and VII)
m_{ij}	Element of Markov transition matrix which denotes the probability that the stock level is j items at the beginning of the subsequent cycle, given that the stock level was i items at the beginning of the present order cycle. (III, IV, V, and VI)
M	Markov transition matrix which is composed of elements, m_{ij}. (III, IV, V, and VI)
N	Minimum number of periods between beginning cycle replenishment orders. (III, IV, VI, VII, and VIII)
ω	Index used for length of cycle. (V and VII)
Ω	Length-of-cycle random variable. (V, VII, and VIII)
$P\{D(\lambda)=\nu\}$	Probability that the demand during λ periods is equal to ν. (III, IV, V, VI, and VII)
$P\{L\}$	Probability of one or more shortages during the order cycle. (VII and VIII)
$P\{X=x\}$	Probability that the lead time random variable is equal to x. (III, IV, V, VI, and VII)
$P\{\Omega=\omega\}$	Probability that the length-of-cycle random variable is equal to ω. (V and VII)
r	Index used for denoting a specific period. (III, IV, V, VI, and VII)
RP	Reorder point at which a replenishment order is placed at the beginning of the first period in which the stock level is equal to or below this point. (V, VI, VII, and VIII)
s	Lower order level which is used with the (s,S) inventory policy. A replenishment order is placed if the beginning stock level is equal to or less than s. (IV, VII, and VIII)
S	Order level which equals the sum of stock on hand and stock on order after the replenishment order is placed. (III, IV, V, VI, VII, and VIII)

x	Index used for a specific value of replenishment lead time. (III, IV, V, VI, and VII)
X	Replenishment lead time random variable. (III, IV, V, VI, and VII)

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VITA

Thomas L. Newberry, Jr. was born in Glasgow, Kentucky, on January 23, 1933, the son of Thomas and Blanche (nee Hall) Newberry. He attended public schools in Russellville, Kentucky. In May of 1950, he was graduated from Kentucky Military Institute in Lyndon, Kentucky.

In September of 1950, the author entered Georgia Institute of Technology. He was a member of the Alpha Tau Omega social fraternity. While at Georgia Tech his memberships included Alpha Pi Mu, Tau Beta Pi, Phi Kappa Phi, Omicron Delta Kappa, and ANAK. In addition, he was editor of the Georgia Tech Engineer. He was a recipient of a General Electric scholarship, and during the summer of 1953 he worked as an industrial engineer for the General Electric Company in Lynn, Massachusetts. On July 25, 1953, the author was married to Evelyn Mason. In March, 1954, he was graduated from Georgia Institute of Technology with the degree of Bachelor of Industrial Engineering.

Following graduation he worked as a consulting engineer for four weeks with a small manufacturing company in Atlanta. After completing this work, he was employed by Phillips Petroleum Company as an industrial engineer until he entered the Armed Forces.

On June 13, 1954, he entered the United States Air Force as a second lieutenant. After spending several months in electronic schools and special weapons schools, the author was stationed at the Nevada Test Site. His last two years in the Air Force were spent at Rushmore Air Force Station with the rank of first lieutenant as Production Control

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Returning to civilian life, the author was employed on a half-time basis as an instructor in the School of Industrial Engineering at Georgia Institute of Technology, while he was pursuing studies for the master's degree in industrial engineering. He received the degree Master of Science in Industrial Engineering in June, 1958. He was elected to the Sigma Xi honorary society in June, 1958. The topic of his master's thesis was The Development of an Electronic Analog Model for the Study of the Economic System of the United States, under the direction of Dr. Joseph Krol.

In June, 1958, he entered the Ph.D. program in the School of Industrial Engineering at Georgia Institute of Technology. On December 4, 1948, a second daughter, Suzanne Blanche, was born. In July, 1959, the author transferred from teaching in the School of Industrial Engineering to part-time work in the Engineering Experiment Station. During this time he worked as an industrial engineer on a project, under the direction of Dr. Harold Smalley, investigating the relative economic feasibility of disposable versus reprocessed supply items for hospitals. Also the author worked as an Assistant Research Engineer with the Rich Electronic Computer Center as a consultant to industrial firms for computer application of statistics, inventory, and production problems. In January, 1960, he was registered as a professional engineer in the State of Georgia. A third daughter, Jennifer Lynn, was born July 7, 1960. This work at the Engineering Experiment Station continued until April, 1961.

During the period June, 1957, through April, 1961, the author was engaged as a consulting engineer with various industrial firms. Some of the more significant projects are summarized as follows:

planning and direction of parts provisioning study for a major airline; development of plant layout and materials handling system for two medium-sized manufacturing plants; analysis of pricing policy for a multi-product, medium-sized manufacturing firm and for the operating room and recovery room in a medium-sized hospital; development of standards for manning reservation offices and ticket offices in a multi-office transportation firm, for cleaning operations in a multi-store retailing firm, and for processing clerical work in a large central clearing agency; development of a scheduling system for operating rooms in a medium-sized hospital and for routine reports in a medium-sized data processing section.

Also, during this period, the author was either author or co-author of the following publications:

"The Error of Estimate in Systematic Activity Sampling," Journal of Industrial Engineering, July-August 1960, p. 290 (with H. O. Davidson and W. W. Hines).

"A Classification of Inventory Control," Journal of Industrial Engineering, September-October 1960, p. 391.

"Inventory Reorder Points for Conditions of Variable Demand and Lead Time," Journal of Industrial Engineering, January-February 1961, p. 32 (with J. B. Talbird and D. C. Ekey).

"Profit Maximization and Industrial Engineering," Journal of Industrial Engineering, May-June 1961 (with H. E. Smalley).

After completing the substantial portion of his Ph.D. thesis, the author joined on April 10, 1961, Operations Research Incorporated as Manager and Technical Director, Atlanta Office.